MULTIPLIERS OF Ap-SPACES, THE CLASS L(LOG L) $^{\alpha}$ AND SOME LACUNARY SETS

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TO MY

MOTHER

CERTIFICATE

This is to certify that the work embodied in the thesis ''Multipliers of A_p-spaces, the Class $L(\mathrm{Log}\ L)^{\alpha}$ and some lacunary sets'' by Sanjiv Kumar Gupta has been carried out under my supervision and has not been submitted elsewhere for a degree

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CONTENTS

CHAPTER I

INTRODUCTION

Larsen, Liu and Wang [20] introduced the algebras $A_p(G)$ $(1 \le p < \infty)$ of integrable functions on a locally compact abelian group G with Fourier transforms in $L^p(\Gamma)$ These algebras and their multipliers were studied by several authors, namely, Figa-Talamanca and Gaudry [8], Martin and Yap [22], Reiter [25], Tewari and Gupta [29, 30] and Bloom [4]. These algebras are similar to group algebra in many ways and are particular examples of Segal algebras [25] All Segal algebras are commutative semi-simple Banach algebras and that their maximal ideal space can be identified with Γ [25] A complete characterization of the multiplier space (A_p, A_q) is not known for different values of p and q. For a non-compact group $(A_p, A_p) = \hat{M}(G)$ [8] For some partial results on (A_p, A_q) multiplier spaces we refer to [4], [18], [30].

In this thesis, we study several problems related to the multipliers from A_p to A_q. We briefly describe the contents of various chapters

In the 2nd chapter we set our notation, give basic definitions and state well-known results which are needed later.

Before we describe the contents of Chapter III, we need the following definition.

<u>Definition</u>: Let S_1 and S_2 be two subsets of $L^1(G)$ and Φ a bounded function on Γ. Then Φ is said to be a multiplier from S_1 to S_2 if $\Phi \hat{f} \in \hat{S}_2 \ \forall f \in \hat{S}_1$

In Chapter III, we study the multiplier spaces (A_p , A_q) for a compact abelian group If $1 \le p \le 2$, then, using Plancherel theorem, it is easy to see that $(A_p , A_p) = \ell_\infty$. For $2 < p < \infty$, the spaces (A_p , A_p) have not been characterized However, it is known that for different values of p these spaces are distinct In fact, some stronger results are known [30], namely,

(i) If $2 < q < p < \infty$, then $(A_p , A_p) \cap C_0(\Gamma) \subsetneq (A_q , A_q) \cap C_0(\Gamma)$

(ii) $\bigcup_{2<q<p} (A_p , A_p) \subsetneq (A_q , A_q)$

In [28], the proper containment of $(A_p \ \ A_p)$ in $\bigcap_{2<q<p}(A_q , A_q)$ was mentioned as an open problem We prove this in the 2nd section. The proof of this result uses an interesting lemma about the sequence spaces ℓ_p , which we again use to improve some results on (A_p , A_q) multipliers due to Tewari and Gupta [30].

In the next section we consider the permutation invariant multipliers from A_p to A_q for different values of p,q. To state our results, we need the following definitions

<u>Definition</u>: Let Γ be a discrete abelian group We say that π is a permutation of Γ if π is one-one onto mapping of Γ

<u>Definition</u>: Let S_1 and S_2 be two subsets of $L^1(G)$ and Φ a function defined on Γ We say Φ is a permutation invariant multiplier from

S_1 to S_2 if $\phi o\pi \in (S_1 , S_2)$ for every permutation π of Γ The set of permutation invariant multipliers is denoted by $\Pi(S_1 , S_2)$

The following are the main results of this section : Let $G=\mathbb{T}$, then

(A) $\Pi(A_r(\mathbb{T}) , A_q(\mathbb{T})) = \ell_{\frac{rq}{r-q}}(Z)$, $1 \le q \le 2 < r < \infty$.

(B) Let $r > 2$, and ϕ be a complex-valued function defined on Z such that $\varepsilon\ \phi \in \Pi(A_r(\mathbb{T}) ,A_r(\mathbb{T}))$, for every function ε defined on Z whose range is contained in $\{\pm 1\}$, then $\phi \in \ell_{\frac{2r}{r-2}}(Z)$ (Note that $1 \in \Pi(A_r(\mathbb{T}) ,A_r(\mathbb{T}))$, but $1 \notin \ell_{\frac{2r}{r-2}}(Z)$).

In section 4, we give a sufficient condition on $\phi \in (A_r(\mathbb{T}), \ell_q(Z)$, $1 \le q < r < \alpha$, $r > 2$ so that $\phi \in \ell_{\frac{rq}{r-q}}(Z)$

Chapter IV contains five sections. In section 2 we study multipliers from L^p to ℓ_q These multipliers have been studied in [11] The following results follow from well-known results Let G be a compact abelian group, then

(α) For $p \ge 2$, $(L^p , \ell_q) = \ell_{\frac{2q}{2-q}}$, $1 \le q < 2$

$= \ell_\infty$, $q \ge 2$.

(β) For $1 < p \le 2$, $(L^p , \ell_q) = \ell_\infty$, $q \ge p'$

(γ) $(L^1 , \ell_q) = \ell_q$, $1 \le q \le \infty$.

In view of (α), (β), (γ) the study of (L^p , ℓ_q)-multipliers is interesting only when $1 < p < 2$ and $1 \le q < p'$

The main results of this section are .

(a) Suppose G is not totally disconnected and $1 \le p_1 < p_2 \le 2$ Then

$$(L^{p_1}, \ell_q) \subsetneq (L^{p_2}, \ell_q) , \quad 1 \le q < p_1'$$

(b) Let G be a compact abelian group then for $1 < p < 2$, $1 \le q < p'$,
$$\ell_{\frac{p'q}{p'-q}} \subsetneq (L^p , \ell_q)$$

To describe the contents of section 3, we introduce some notation. Let S be a subset of L^1 and $1 \le p < \infty$. Define

$$S_p = \{f \in S | \hat{f} \in \ell_p\}$$

Suppose $p > 1$, then the following containments are not hard to see.

$$\ell_{\frac{rq}{r-q}} \subseteq (L^p_r , \ell_q) , \quad 1 \le q < r < \infty, \ r > 2.$$

$$\hat{M}(G) + \ell_{\frac{2r}{r-2}} \subseteq (L^p_r , A_r) , \quad r > 2$$

$$\hat{M}_{\frac{qr}{r-q}}(G) + \ell_{\frac{2r}{r-2}} \subseteq (L^p_r , A_q) , \quad 2 < q < r < \infty.$$

In this section, we show that each of these inclusions is proper.

In section 4, we study permutation invariant multipliers from L^p_r to ℓ_q and prove the following results

(A) Suppose $r > 2$, $1 \le q < r < \infty$ and $1 \le p < r'$. Then

$$\Pi(L^p_r(\mathbb{T}) , \ell_q(Z)) = \ell_{\frac{rq}{r-q}}(Z)$$

Also there exists a permutation invariant multiplier in $(L^{r'}(\mathbb{T}) , \ell_q(Z))$ which is not in $\ell_{\frac{rq}{r-q}}(Z)$.

(B) Let $r > 2$, and ϕ be a complex valued function defined on Z such that $\varepsilon\,\phi \in \Pi(L^p_r(\mathbb{T})\ ,\ A_r(\mathbb{T}))$ for every function ε defined on Z whose range is contained in $\{\pm 1\}$, then $\phi \in \ell_{\frac{2r}{r-2}}(Z)$ (Note that $1 \in \Pi(L^p_r(\mathbb{T})\ ,\ A_r(\mathbb{T}))$ but $1 \notin \ell_{\frac{2r}{r-2}}(Z)$)

Suppose $r > 2$, $1 \le q < r < \infty$ and $1 \le p < r'$ In section 5, we give a sufficient condition on $\phi \in (L^p_r(\mathbb{T})\ ,\ \ell_q(Z))$ so that $\phi \in \ell_{\frac{rq}{r-q}}(Z)$.

In Chapter V, we use Rudin-Shapiro polynomials to study the conjugation problem in A_p-spaces. In the second section, we describe the construction of Rudin-Shapiro polynomials on a general nondiscrete locally compact abelian group, as given in [11].

In section 3, we study the problem of conjugation in the spaces $A_p(G)$ for a compact connected abelian group G. We observe that

(i) $A_p \subseteq L^2$, $1 \le p \le 2$ and (ii) $L^{p'} \subseteq A_p \subseteq L^1$ when $p > 2$

It is known that L^p admits conjugation when $P > 1$ and L^1 does not admit conjugation. Hence by (i) we need to study the conjugation problem in A_p spaces only when $p > 2$. Also, A_p admits conjugation if and only if the Hilbert transform defines a multiplier from $(A_p\ ,\ A_p) = (A_p\ ,\ L^1)$. To state our results, we need some definitions.

<u>Definition</u>: A function Φ defined on $[0,\infty)$ is said to be a Young's function if it is increasing,, continuous, convex, and satisfies

$$(1) \quad \lim_{t\to 0} \frac{\Phi(t)}{t} = 0 \qquad \text{and}$$

$$(2) \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$$

Let Φ be a Young's function. If f is a measurable function on G, we define

$$N_\Phi(f) = \int_G \Phi(|f(x)|)dx \ ,$$

where $N_\Phi(f)$ is defined to be ∞, if $\Phi \circ |f|$ is not integrable. The Orlicz space $L^\Phi(G)$ is defined as follows

$L^\Phi(G)=\{f \ . \ f$ is measurable and $N_\Phi(\lambda f) < \infty$ for some $\lambda > 0\}$

For $f \in L^\Phi$, we define

$$||f||_{L^\Phi} = \inf_{\lambda > 0} \ 1/\lambda \ \{N_\Phi(\lambda f) + 1\}$$

Then $(L^\Phi, || \ \ ||_\Phi)$ is a Banach space.

Also, it is clear that $L^\Phi \subsetneqq L^1$ for every Young's function, so that

$$(A_p \ , L^\Phi) \subseteq (A_p \ , L^1)$$

The main result of this section is the following:

Let G be a compact connected abelian group and $p > 2$ Then the Hilbert transform does not belong to $(A_p \ , L^\Phi)$ for any Young's function Φ

In the last section, we show that if G is a nondiscrete locally compact abelian group and $p > 2$, then $A_p(G)$ is not contained in $L^\Phi(G)$, for any Young's function Φ The proof of this result uses the methods of section 3. As a corollary of this theorem, we get a result of H.C Lai [15]

To describe the contents of Chapter VI we observe that

$$(A_p , A_q) \subseteq (S_p , A_q)$$

when S is a subset of L^1

Let $B_\alpha(G) = L(\log^+ L)^\alpha$ $\alpha > 0$ and $B_{\alpha,r} = \{f \in B_\alpha | \hat{f} \in \ell_r\}$ Then for $p > 1$ and $\alpha > 0$, we have

$$\begin{array}{lccccccc} & \mathrm{I} & & \mathrm{II} & & \mathrm{III} & & \mathrm{IV} \\ (i) & \ell_{\frac{rq}{r-q}} & \subseteq & (A_r , \ell_q) & \subseteq & (B_{\alpha,r} , \ell_q) & \subseteq & (L^p_r , \ell_q) \end{array}$$

when $1 \le q < r < \infty$, $r > 2$

$$(ii) \quad \hat{M}(G) + \ell_{\frac{2r}{r-2}} \subseteq (A_r , A_r) \subseteq (B_{\alpha,r} , A_r) \subseteq (L^p_r , A_r)$$

when $r > 2$.

$$(iii) \quad \hat{M}_{\frac{rq}{r-q}}(G) + \ell_{\frac{2r}{r-2}} \subseteq (A_r , A_q) \subseteq (B_{\alpha,r} , A_q) \subseteq (L^p_r , A_q)$$

when $2 < q < r < \infty$

The proper inclusion of I in II is not known and we have proved the proper inclusion of I in IV in Chapter IV In that proof, we use the fact that there exists an infinite subset E of Γ such that $\hat{L}^p|_E \subseteq \ell_2(E)$ (any Sidon subset of Γ has this property). Zygmund [33] showed that for any Hadamard subset E of Z, $\hat{B}^{1/2}|_E \subseteq \ell_2(E)$ Using this, one easily sees that if $G = \mathbb{T}$ and $\alpha \ge 1/2$ then the inclusion of I in III is proper in each of the cases (i)-(iii) These results inspired us to investigate similar results for the classes B_α on an arbitrary compact abelian group G and Sidon subsets E of Γ In section 2, as a generalization of Zygmund's result, we show that if E is a Sidon

subset of Γ and $0 < \alpha \le 1/2$ then $\hat{B}_\alpha|_E \subseteq \ell_{1/\alpha}(E)$ and that there exists a Sidon subset E of Γ such that $\hat{B}_\alpha|_E \nsubseteq \ell_r(E)$ for any $r < 1/\alpha$

Using the above mentioned result we prove that if $\alpha \ge 1/2$, then the inclusion of I in III is proper in each of the cases (i)-(iii) Further if $\alpha < 1/2$, $1/\alpha < r < \infty$, and $1 \le q < r$, we prove that

$$\ell_{\frac{rq}{r-q}} \subsetneq (B_{\alpha,r} , \ell_q)$$

Next we considered the problem of the existence of non-Sidon sets E for which $\hat{B}_\alpha|_E \subseteq \ell_{1/\alpha}(E)$ If E is an infinite Sidon set, then E+E is a non-Sidon set,and for a particular Sidon subset E of Z ($E = \{2^k\}_{k\in N}$) we found that if $E_k = E + E + \quad +E$(k-times), then $\hat{B}_\alpha|_{E_k} \subseteq \ell_{k/\alpha}(E_k)$ for $0 < \alpha \le k/2$ and $\hat{B}_\alpha|_{E_k} \nsubseteq \ell_r(E_k)$, $r < k/\alpha$ These considerations led us to define and study a new class of lacunary sets, which we call $\Lambda_{\alpha,\beta}$ sets. These are defined as : A subset $E \subseteq \Gamma$ is called a $\Lambda_{\alpha,\beta}$ set if $\hat{B}_\alpha|_E \subseteq \ell_{2\beta/\alpha}(E)$, $0 < \alpha \le \beta$.

In view of this definition, the above mentioned result of Zygmund states that a Hadamard set of positive integers is a $\Lambda_{1/2,1/2}$ set, and our generalization of the result states that a Sidon subset of Γ is a $\Lambda_{\alpha,1/2}$ set, where $0 < \alpha \le 1/2$

In section 2, we give some characterizations of $\Lambda_{\alpha,\beta}$ sets. As a corollary, using a result of Pisier [23], we get a new characterization of Sidon sets We also give some examples of $\Lambda_{\alpha,\beta}$ sets in this section

It is easy to see that if $\beta_1 < \beta_2$ then every Λ_{α,β_1} set is also a Λ_{α,β_2} set It is natural to ask whether the class of $\Lambda_{\alpha\ \beta_1}$ sets is a proper subclass of Λ_{α,β_2} sets In section 3, we show that for each positive integer k there exists a subset $E \subseteq \Gamma$ which is a $\Lambda_{\alpha,k/2}$ set for $0 < \alpha \leq k/2$ but not a $\Lambda_{\alpha,\beta}$ set for any $\beta < k/2$

CHAPTER II

NOTATION AND PRELIMINARIES

In this chapter we specify the notation and state standard results which are needed in later chapters.

1 1 Notation. All the groups which we consider will be locally compact abelian groups. Every linear space considered will be over $\mathbb{C}$, the field of complex numbers

If G is a locally compact abelian group, then Γ will denote the dual group Integration on G with respect to Haar measure is denoted by dx For $1 \le p < \infty$, $L^p(G)$ (or just L^p) denotes the linear space of equivalence classes of complex-valued measurable functions on G whose p^{th} power is integrable with respect to the Haar measure The linear space of equivalence classes of essentially bounded complex-valued measurable functions on G will be denoted by $L^\infty(G)$. $L^p(G)$ is a Banach space under the norm

$$||f||_{L^p} = (\int_G |f(x)|^p \, dx)^{1/p} \qquad (1 \le p < \infty)$$

$$||f||_{L^\infty} = \operatorname*{ess\,sup}_{x \in G} |f(x)| \qquad (p = \infty)$$

$C(G)$, $C_o(G)$ and $C_c(G)$ denote, respectively, the spaces of continuous complex-valued functions on G that are bounded, vanish at infinity, or have compact support.

The spaces $C(G)$ and $C_o(G)$ are Banach spaces with the norm

$$||f||_\infty = \sup_{x\in G} |f(x)|$$

The space $L^1(G)$ is a commutative Banach algebra with the convolution

$$f*g(x) = \int_G f(x-y)\, g(y)\, dy,$$

as a multiplication operation The maximal ideal space of $L^1(G)$ can be identified with Γ The Fourier transform $\hat{f}$ of f in $L^1(G)$ is given by

$$\hat{f}(\gamma) = \int_G (-x,\gamma)\, f(x)\, dx \qquad (\gamma \in \Gamma)$$

If G is a compact abelian group, then $L^p(G)$, $1 \le p < \infty$, is also a commutative Banach algebra with convolution as multiplication. The maximal ideal space can again be identified with Γ

If G is discrete we denote $L^p(G)$ by $\ell_p(G)$, or simply ℓ_p

The space of all bounded, complex-valued regular Borel measures on G is denoted by $M(G)$ $M(G)$ is a Banach space with the norm $||\mu|| = |\mu|(G)$ where $|\mu|$ is the total variation of μ. $M(G)$ is also a commutative Banach algebra under the convolution defined as

$$\mu*\nu(E) = \int_G \mu(E-x)\, d\nu(x),$$

If f is a function on G, and $x \in G$ the translate $\tau_x f$ is defined by $\tau_x f(y) = f(y-x)$ for every $y \in G$

Let H be a closed subgroup of G. Suppose $f \in L^1(G)$, define

$$\Pi_H(f) = \int_H f(x+y)\, dm_H(y)$$

where m_H denotes the Haar measure on H Then Π_H maps $L^1(G)$ onto $L^1(G/H)$ We shall use the same notation Π_H for the quotient mapping from G onto G/H.

Let E a subset of G For a function f on G, by $f|_E$ we denote the restriction of the function f to E. For a subset S of $L^1(G)$ by $S|_E$ we denote the set $\{f|_E| \ f \in S\}$

Let S be a subset of $L^1(G)$ and E a subset of Γ. By S_E we denote, the set $\{f \in S| \ \hat{f} = 0 \text{ on } \Gamma\backslash E\}$

Let $1 < r < \infty$ By r' we denote the conjugate index of r, i e $\frac{1}{r} + \frac{1}{r'} = 1$ The cardinality of a set E is written as $|E|$. χ_E denotes the characteristic function of the set E By E-E we denote the set $\{x-y| \ x \in E, \ y \in E\}$ C, C_1, C_2 are constants which may vary from one line to the next The space of trigonometric polynomials on G is denoted by $\mathcal{T}$. $\mathbb{T}$ denotes the circle group. For other notation we refer to Hewitt and Ross [10, 11] and Lopez and Ross [21]

1.2 <u>Basic properties of $A_p(G)$</u>: If G is a locally compact abelian group and $1 \le p \le \infty$, we define

$$A_p(G) = \{f| \ f \in L^1(G), \ \hat{f} \in L^p(\Gamma)\}$$

We often write $A_p(G)$ as A_p These spaces were introduced by Larsen, Liu and Wang [20] If $p = \infty$ or if G is discrete, then it is clear that $A_p(G) = L^1(G)$. Therefore, we shall always assume that G is nondiscrete and $1 \le p < \infty$.

If $1 \le p < \infty$, then with the norm

$$||f||_{A_p} = ||f||_{L^1} + ||\hat{f}||_{L^p} \quad (f \in A_p(G))$$

$A_p(G)$ is a commutative Banach algebra with convolution as multiplication. The maximal ideal space of $A_p(G)$ can be identified with Γ [18].

It is clear that $A_p(G) \subseteq A_q(G)$ when $1 \le p < q < \infty$ If G is nondiscrete then Tewari and Gupta [29] proved that $A_p(G) \subsetneq A_q(G)$.

For results on A_p spaces and further references, see the survey article by Larsen [17].

1.3 Segal Algebras: Let G be an arbitrary locally compact abelian group. A linear subspace S of $L^1(G)$ is said to be a Segal algebra, if it satisfies the following conditions

(i) S is dense in $L^1(G)$

(ii) S is a Banach space under a norm $\|\ \|_S$ such that

$$\|f\|_S \ge \|f\|_{L^1} \quad \forall\ f \in S$$

(iii) If $f \in S$ then $\tau_x f \in S$ $\forall\ x \in G$ and $\|\tau_x f\|_S = \|f\|_S$

(iv) For each $f \in S$ the mapping $x \longrightarrow \tau_x f$ of G into S is continuous.

The spaces L^1, $L^1 \cap L^p$ $(1 < p < \infty)$ $L^1 \cap C_o(G)$, $A_p(1 \le p < \infty)$ are examples of Segal algebras.

The set of functions whose Fourier transforms have compact support is dense in every Segal algebra [25]

1.4 Multipliers:

(a) Let G be a locally compact abelian group and S_1 and S_2 two subsets of $L^1(G)$ A bounded function ϕ on Γ is said to be a multiplier from S_1 to S_2 if $\phi\hat{f} \in \hat{S}_2$ for every $f \in S_1$ The set of multipliers from S_1 to S_2 is denoted by (S_1, S_2).

If S_1 and S_2 are Segal algebras, then it is easy to see that a multiplier $\phi \in (S_1, S_2)$ induces a bounded linear operator T from S_1 to S_2 defined by $(Tf)^\wedge = \phi\hat{f}$ $\forall\ f \in S_1$

For basic facts and results on multipliers we refer to [11] and [18]

(b) Let G be a locally compact abelian group and T a linear operator from $L^p(G)$ into $L^q(G)$, $1 \le p,q \le \infty$. We call T a pointwise multiplier if there exists a measurable function ϕ on G such that $Tf = \phi f \quad \forall f \in L^p(G)$. The set of all measurable functions corresponding to pointwise multipliers from $L^p(G)$ into $L^q(G)$ will be denoted by $M(L^p,L^q)$. We shall need the following theorem about pointwise multipliers

<u>Theorem 1.5 ([4])</u>: Let G be a discrete abelian group Then

(i) $M(\ell_p,\ell_q) = \ell_\infty(G)$, $\quad 1 \le p \le q \le \infty$

(ii) $M(\ell_p,\ell_q) = \ell_{\frac{pq}{p-q}}$, $\quad 1 \le q < p < \infty$

1.6 <u>Orlicz Spaces ([14])</u>: Let G be a locally compact abelian group. We first define a Young's function.

<u>Definition</u>: A function ϕ defined on $[0,\infty)$ is said to be a Young's function if it is increasing, continuous, convex, and satisfies

$$(1) \quad \lim_{t\to 0} \frac{\phi(t)}{t} = 0 \qquad \text{and}$$

$$(2) \quad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty$$

Let ϕ be a Young's function If f is a measurable function on G, we define

$$N_\phi(f) = \int_G \phi(|f(x)|)dx \ ,$$

where $N_\phi(f)$ is defined to be ∞, if $\phi \circ |f|$ is not integrable

The Orlicz space $L^{\phi}(G)$ is defined as follows .

$$L^{\phi}(G)=\{f \,.\, f \text{ is measurable and } N_{\phi}(\lambda f) < \infty \text{ for some } \lambda > 0\}$$

We note that L^1 is not an Orlicz space

For $f \in L^{\phi}$, we define

$$||f||_{L^{\phi}} = \inf_{\lambda>0} 1/\lambda \{N_{\phi}(\lambda f) + 1\}$$

Then (L^{ϕ}, $||\ \ ||_{\phi}$)is a Banach space

For the following definitions and results, we assume G to be a compact abelian group

Two Young's functions ϕ_1 and ϕ_2 are equivalent if there exist positive constants k_1, k_2 such that

$$\phi_1(k_1 t) \le \phi_2(t) \le \phi_1(k_2 t) , \qquad t \ge t_o$$

If ϕ_1 and ϕ_2 are two equivalent Young's functions then $L^{\phi_1} = L^{\phi_2}$ and the norms defined by ϕ_1 and ϕ_2 are equivalent

We say that a Young's function ϕ satisfies the Δ_2-condition if there exists a constant $C > 0$ and $t_o \ge 0$ such that

$$\phi(2t) \le C\, \phi(t) \qquad \text{for all } t \ge t_o$$

If ϕ satisfies the Δ_2-condition then the dual of L^{ϕ} is isomorphic to an Orlicz space L^{ψ}, where ψ is also a Young's function given by

$$\psi(s) = \sup_{t\ge 0} (st - \phi(t)) \quad , \ s \ge 0$$

We shall be particularly concerned with the following Young's function Let $\alpha > 0$, we define

$$\phi_\alpha(t) = \begin{cases} \dfrac{t^{1+\alpha}}{e^{\alpha}} & , \ 0 \le t \le e \\ t\,(\log t)^{\alpha} & , \ t > e \end{cases}$$

It is easy to see that ϕ_α is a Young's function satisfying the Δ_2-condition. In fact,

$$\phi_\alpha(2t) \le 2^{1+\alpha}\,\phi_\alpha(t) \qquad \text{for all } t \ge e$$

It is not difficult to show that the dual space $(L^{\phi_\alpha})^*$ is given by L^{ψ_α}, where

$$\psi_\alpha(t) = \begin{cases} \dfrac{e^{2\alpha}\, t^2}{(2\alpha)^{2\alpha}}, & 0 \le t \le (2\alpha)^\alpha \\ e^{t^{1/\alpha}}, & t > (2\alpha)^\alpha \end{cases}$$

1.7 Lacunary sets ([21]).

(a) Sidon sets: An infinite subset E of Γ is called a Sidon set if there exists a constant $C > 0$ such that to each $\phi \in \ell_\infty(E)$, there corresponds a $\mu \in M(G)$ satisfying

$$\phi = \hat{\mu}|_E \quad \text{and} \quad ||\mu|| \le C\, ||\phi||_{\ell_\infty}$$

Any constant C with this property is called a Sidon constant for the set E. We shall need the following theorem about Sidon sets.

Theorem 1.8 ([21]). Let G be a compact abelian group and Γ its dual group. Let E be a Sidon subset of Γ and $1 < p < 2$. Then

$$\hat{L}^p|_E \subseteq \ell_2(E).$$

(b) Λ_p-sets. Let $E \subseteq \Gamma$ and $1 < p < \infty$. E is said to be a Λ_p-set if $L^1_E = L^p_E$. It is known that every Sidon subset of Γ is a Λ_p-set for every p. Also, there exists a subset which is a Λ_p-set for every p and which is not a Sidon set [27].

1.9 In Chapter III, we make frequent use of the following result due to Edwards:

<u>Theorem 1.10 ([6]) or ([11])</u>: Let G be a compact abelian group and ϕ a function defined on Γ. Let $\varepsilon\phi \in \hat{L}^1$ for every function ε defined on Γ whose range is contained in $\{\pm 1\}$, then $\phi \in \ell_2$

CHAPTER III

MULTIPLIERS OF $A_p(G)$

§ 1. INTRODUCTION

In this chapter we study the multiplier spaces $(A_p \ \ A_r)$ for a compact abelian group These spaces have been studied by several authors in a series of papers In particular we refer to Larsen [19], Lai [16], Tewari and Gupta [30], Tewari [28] and Bloom [4] We remark that if G is a non-compact locally compact abelian group, then it is known [8] that

$$(A_p \ , \ A_p) = \hat{M}(G)$$

If $1 \le p \le 2$, using Plancherel theorem it is easy to see that $(A_p \ , \ A_p) = \ell_\infty$ Thus the study of the multiplier spaces $(A_p \ , \ A_p)$ is interesting only when $p > 2$

It follows from the definitions that $(A_p \ , \ A_p) \subseteq (A_q \ . \ A_q)$ when $1 \le q < p < \infty$ Tewari and Gupta [30] proved in 1978 that if $2 < q < p < \infty$, then

$$(A_p \ , \ A_p) \cap C_o(\Gamma) \subsetneq (A_q \ , \ A_q) \cap C_o(\Gamma)$$

and
$$\bigcup_{2<q<p}(A_p \ , \ A_p) \subsetneq (A_q \ , \ A_q)$$

However, the problem of the proper containment of $(A_p \ , \ A_p)$ in $\bigcap_{2<q<p} (A_q \ , \ A_q)$ remained unresolved In section 2 of this chapter we prove that this containment is proper Further, using the ideas

of this proof, we improve some results on (A_p, A_r)-multipliers proved by Tewari and Gupta in [30].

Hewitt and Ross [11] proved that if G is a compact abelian group and $1 \le p \le 2$, then $M(\ell_\infty, \hat{L}^p) = \ell_2$ As a consequence of this result we get the following result

Let π be a permutation of Γ, $1 \le p \le 2$ and ϕ a multiplier in $M(\ell_\infty, \hat{L}^p)$ Then $\phi o \pi$ belongs to the multiplier space $M(\ell_\infty, \hat{L}^p)$

On the other hand, Figa-Talamanca [7] proved in 1965 that if $2<p<\infty$ then there exists a permutation π of Γ and a multiplier ϕ in $M(\ell_\infty, \hat{L}^p)$ such that $\phi o \pi$ does not belong to $M(\ell_\infty, \hat{L}^p)$

In section 3, we study the analogues of the above results for (A_p, A_r)-multipliers First we observe that if $1 \le r \le 2 < p < \infty$ then

$$\ell_{\frac{pr}{p-r}} \subseteq (A_p, A_r)$$

For, if $f \in A_p$ then using Holder's inequality we see that $\phi \hat{f} \in \ell_r$ Since $1 \le r \le 2$, it follows from Plancherel theorem that there exists a $g \in L^2$ such that $\hat{g} = \phi \hat{f}$ Therefore $\phi \hat{f} \in \hat{A}_r$ and we get that $\ell_{\frac{pr}{p-r}} \subseteq (A_p, A_r)$

In Theorem 3 3, we prove the following result.

Let $1 \le r \le 2 < p < \infty$ Then the set of permutation invariant multipliers from $A_p(\mathbb{T})$ to $A_r(\mathbb{T})$ is given by $\ell_{\frac{pr}{p-r}}(Z)$

In section 4, we give a sufficient condition for an $(A_p(\mathbb{T}), A_r(\mathbb{T}))$ multiplier ϕ, $1 \le r \le 2 < p < \infty$, so that ϕ belongs to $\ell_{\frac{pr}{p-r}}(Z)$

§ 2 Proper inclusions in (A_p , A_r)-spaces

In [8] Figa-Talamanca and Gaudry proved that

$$(L^p(G), L^p(G)) \subsetneq (L^q(G), L^q(G)), \qquad (2\ 1)$$

when $1 \le p < q \le 2$ and G is an infinite locally compact abelian group. Price [24] generalized (2 1) by proving

$$\bigcup_{1\le q<p} (L^q(G), L^q(G)) \subsetneq (L^p(G), L^p(G)) \qquad (2\ 2)$$

and

$$(L^p(G), L^p(G)) \subsetneq \bigcap_{p<q\le 2} (L^q(G), L^q(G)) \qquad (2\ 3)$$

when $1 < p < 2$, where the first inclusion is strict when $p = 2$ and the second, when $p = 1$ (Similar results also hold for $2\le p<\infty$)

In view of these results one naturally asks for the corresponding results for (A_p , A_p)-multipliers The following theorem proved by Tewari and Gupta [30] gives the analogues of (2 1)-(2 2) in the case of (A_p , A_p) multipliers

<u>Theorem 2.4 ([30])</u>: Let G be an infinite compact abelian group and $1 \le q < \infty$, $2 < p < \infty$, $p > q$, then

(a) $(A_p , A_p) \cap C_o(\Gamma) \subsetneq (A_q , A_q) \cap C_o(\Gamma)$

(b) $\bigcup_{p>q} (A_p , A_p) \subsetneq (A_q , A_q)$

We state and prove below the analogue of (2 3)

<u>Theorem 2.5</u> : Let G be a compact abelian group Let $p>2$, then

$$(A_p , A_p) \subsetneq \bigcap_{q<p} (A_q , A_q)$$

The proof of Theorem 2 5 depends on the following lemma which may be of independent interest and is suggested by the equality

$$(\ell_{\frac{rp}{p-r}}, \ell_r) = \ell_p, \quad 1 \le r < p < \alpha.$$

<u>Lemma 2 6</u>. Let I be an infinite set Let $1 \le r < p < \alpha$ and $\phi \in \ell_p(I)$ be such that $\phi \notin \ell_q(I)$ for every $q < p$ Then there exists a $\psi \in \bigcap_{t > \frac{rp}{p-r}} \ell_t(I)$ such that $\phi\,\psi \notin \ell_r(I)$.

<u>Proof</u>: Without loss of generality, we assume $|\phi| \le 1$ on I. Let $s > p/r$ be a fixed positive integer Let $q_j = p - 1/j$. Choose $m_o \in \mathbb{N}$ such that $q_j > r$ for $j \ge m_o$ Define, $\alpha_j = \frac{rq_j}{q_j - r}$, $j \ge m_o$

Then q_j increases to p and α_j decreases to $\frac{rp}{p-r}$ Let $(a_n)_{n=1}^{\alpha}$ be the support of ϕ Let $n_o = 0$ and choose $n_1 > 1$ such that

$$\sum_{n=1}^{n_1} |\phi(a_n)|^{q_s} > 1$$

Then choose $n_2 > n_1$ satisfying

$$\sum_{n=n_1+1}^{n_2} |\phi(a_n)|^{q_{2s}} > 1$$

Then define a sequence $(n_j)_{j=1}^{\infty}$ of natural numbers inductively, such that $n_j > n_{j-1}$ $\forall\ j \ge 1$, and

$$\sum_{n=n_{j-1}+1}^{n_j} |\phi(a_n)|^{q_{js}} > 1, \qquad \forall\ j \ge 1$$

We now define ψ as follows .

$$\text{Let} \quad \psi(a) = \begin{cases} |\phi(a_n)|^{p/\alpha_j} & \text{if} \quad a = a_n \ , \ n_{j-1} < n \le n_j \ , \ \forall \ j \ge 1 \\ 0 & \text{on} \quad I\setminus\{a_n\}_{n=1}^{\infty} \end{cases}$$

We claim that ψ satisfies the assertion of the lemma First, we show that $\psi \in \bigcap_{t>\frac{rp}{p-r}} \ell_t(I)$ Note that $\bigcap_{t>\frac{rp}{p-r}} \ell_t(I) = \bigcap_{k=m_o}^{\infty} \ell_{\alpha_k}(I)$.

Hence it suffices to show that $\psi \in \ell_{\alpha_k}(I)$ for every $k \ge m_o$ Let $k \ge m_o$ and consider

$$\sum_{n=n_k+1}^{\infty} |\psi(a_n)|^{\alpha_k} = \sum_{j=k}^{\infty} \sum_{n=n_j+1}^{n_{j+1}} |\phi(a_n)|^{\alpha_k p/\alpha_{j+1}}$$

By the definition, α_k is decreasing. Since $|\phi| \le 1$, we have

$$|\phi|^{\frac{p\alpha_k}{\alpha_{j+1}}} \le |\phi|^p \quad \forall \ j \ge k$$

Hence

$$\sum_{n=n_k+1}^{\infty} |\psi(a_n)|^{\alpha_k} \le \sum_{n=n_k+1}^{\infty} |\phi(a_n)|^p < \infty .$$

Therefore $\psi \in \ell_{\alpha_k}(I)$.

Next we show that $\phi\psi \notin \ell_r(I)$ Consider,

$$\sum_{n=1}^{\infty} |\phi(a_n)|^r |\psi(a_n)|^r = \sum_{j=1}^{\infty} \sum_{n=n_{j-1}+1}^{n_j} |\phi(a_n)|^{pr/\alpha_j} |\phi(a_n)|^r$$

Since $s > p/r$ it is easy to see that

$$r + pr/\alpha_j < q_{js} \qquad \forall \ j \ge 1$$

Hence $\sum_{n=1}^{\infty} |\phi(a_n)|^r |\psi(a_n)|^r \geq \sum_{j=1}^{\infty} \sum_{n=n_{j-1}+1}^{n_j} |\phi(a_n)|^{q_j s} = \infty$,

so that $\phi\psi \notin \ell(I)$. This completes the proof of the lemma

<u>Proof of Theorem 2.5</u>. Since $\bigcup_{q<p} A_q \subsetneq A_p$ [29], there exists an $f \in A_p$ such that $\hat{f} \notin \ell_q$ for every $q < p$ Let $\phi = \hat{f}$ Then ϕ satisfies the conditions of Lemma 2 6 with $r = 2$ Hence there exists $\psi \in \bigcap_{t > \frac{2p}{p-2}} \ell_t$ such that $\phi\psi \notin \ell_2$ For $2 < q < p$, we have $\frac{2q}{q-2} > \frac{2p}{p-2}$, so that $\bigcap_{t>\frac{2p}{p-2}} \ell_t = \bigcap_{p>q>2} \ell_{\frac{2q}{q-2}}$ Hence $\psi \in \bigcap_{p>q>2} \ell_{\frac{2q}{q-2}}$.

We can choose a function ε on Γ whose range is contained in $\{\pm 1\}$ such that $\varepsilon\, \phi\, \psi \notin (L^1)\hat{}$ (see, Theorem 1 10. Ch.II) Then $\varepsilon\, \psi$ belongs to $\bigcap_{p>q>2} \ell_{\frac{2q}{q-2}} \subseteq \bigcap_{p>q>2} (A_q, A_q)$, and, $\varepsilon\, \psi \notin (A_p, A_p)$. This completes the proof of the theorem

For the circle group, we give another proof of Theorem 2 5 which does not use Lemma 2.6

<u>Another Proof of Theorem 2.5 for $G = \mathbb{T}$</u>: The idea of the proof is still the same We construct a $\psi \in \bigcap_{p>q>2} \ell_{\frac{2q}{q-2}}(Z)$ such that $\psi \notin (A_p(\mathbb{T}), A_p(\mathbb{T}))$ Since $\ell_{\frac{2p}{p-2}}(Z) \subsetneq \bigcap_{p>q>2} \ell_{\frac{2q}{q-2}}(Z)$, we choose a $\psi \in \bigcap_{p>q>2} \ell_{\frac{2q}{q-2}}(Z)$ such that $\psi \notin \ell_{\frac{2p}{p-2}}(Z)$ Now $\ell_{\frac{2p}{p-2}}(Z)$ is equal to $M(\ell_p(Z), \ell_2(Z))$. So there exists $\phi \in \ell_p(Z)$ such that $\phi\psi \notin \ell_2(Z)$

Define

$$\phi_1(n) = \max\left(\frac{1}{|n|} + |\phi(n)|, \frac{1}{|n|} + |\phi(-n)|\right)$$

Then $\phi_1 \in \ell_p(Z)$ and $\phi_1\psi \notin \ell_2(Z)$ Also ϕ_1 satisfies the conditions of Lemma 3 4 (see § 3 below) So we can choose a permutation π of Z such that $\phi_1 o\pi \in \hat{L}^1$ Now the proof can be completed exactly as in the general case because $(\phi_1 o\pi)(\psi o\pi) \notin \ell_2(Z)$

2.7 Discussion of (A_p, A_q)-multipliers

First we state some known results

Proposition 2.8 ([30]). Let G be a compact abelian group Then

$$(A_p, A_p) = (A_p, A_q) = \ell_\infty \quad \text{for } 1 \le p \le 2,\ p \le q$$

$$(A_p, A_q) = \ell_{\frac{pq}{p-q}} \quad \text{for } 1 \le q \le p \le 2$$

$$\ell_{\frac{rp}{r-p}} \subseteq (A_p, A_r) \quad \text{for } 1 \le r \le 2 < p < \infty \tag{2 9}$$

$$\hat{M}(G) + \ell_{\frac{2p}{p-2}} \subseteq (A_p, A_p) \quad \text{for } p > 2. \tag{2 10}$$

$$\hat{M}_{\frac{rp}{p-r}}(G) + \ell_{\frac{2p}{p-2}} \subseteq (A_p, A_r) \text{ for } 2 < r < p < \infty. \tag{2.11}$$

It is not known whether the inclusions in (2 9)-(2.11) are proper. However Tewari and Gupta [30] proved the following proposition

Proposition 2.12 ([30]): Let G be a compact abelian group Then

$$\ell_s \nsubseteq (A_p, A_r) \quad \text{when } 1 \le r \le 2 < p < \infty \text{ and } s > \frac{rp}{p-r}$$

$$\ell_s \nsubseteq (A_p, A_p) \quad \text{when } p > 2 \text{ and } s > 2p/(p-2)$$

$$\ell_s \nsubseteq (A_p, A_r) \quad \text{when } 2 < r < p < \infty \text{ and } s > \frac{2p}{p-2}$$

The proof of Proposition 2 12 in [30] fails to show the following stronger results

$$\bigcap_{s>\frac{rp}{r-p}} \ell_s \nsubseteq (A_p , A_r), \ 1 \le r \le 2 < p < \infty \tag{2.13}$$

$$\bigcap_{s>\frac{2p}{p-2}} \ell_s \nsubseteq (A_p , A_p), \ p > 2 \tag{2 14}$$

$$\bigcap_{s>\frac{2p}{p-2}} \ell_s \nsubseteq (A_p , A_r), \ 2 < r < p < \alpha \tag{2 15}$$

We prove (2 13)-(2 15) using the ideas of the proof of Theorem 2.5

<u>Theorem 2 16</u>·Let G be an infinite compact abelian group Then

(a) $\bigcap_{s>\frac{rp}{p-r}} \ell_s \nsubseteq (A_p , A_r), \ 1 \le r \le 2 \le p < \infty$

(b) $\bigcap_{s>\frac{2p}{p-2}} \ell_s \nsubseteq (A_p , A_p), \ p > 2.$

(c) $\bigcap_{s>\frac{2p}{p-2}} \ell_s \nsubseteq (A_p , A_r), \ 2 < r < p < \infty$

<u>Proof</u>:(a) Since $\bigcup_{q<p} A_q \subsetneq A_p$, there exists an $f \in A_p$ such that $\hat{f} \notin \ell_q$ for every $q < p$. Hence by Lemma 2.6 we get $\psi \in \bigcap_{s>\frac{rp}{p-r}} \ell_s$ such that $\psi \hat{f} \notin \ell_r$ Thus $\psi \notin (A_p , A_r)$.

(b) Using (a) for r=2 we get $\Phi \in \bigcap_{s>\frac{2p}{p-2}} \ell_s$ such that $\Phi \notin (A_p , A_2)$. Hence there exists an $f \in A_p$ such that $\Phi \hat{f} \notin \ell_2$.Now there exists a function ε defined on Γ with range in $\{\pm 1\}$ such that $\varepsilon \Phi \hat{f} \notin (L^1\hat{)}$. Hence $\varepsilon \Phi \notin (A_p , A_p)$ and $\varepsilon \Phi \in \bigcap_{s>\frac{2p}{p-2}} \ell_s$.

(c) follows from (b).

§3. Permutation invariant multipliers from A_p to A_r

We begin with some definitions·

Definition 3.1. Let Γ be a discrete abelian group We say that π is a permutation of Γ if π is a one-one onto mapping of Γ

Definition 3 2: Let G be a compact abelian group and Γ its dual group Let Φ be a function defined on Γ. We say that Φ is a permutation invariant multiplier from A_p to A_r if $\Phi o \pi \in (A_p , A_r)$ for every permutation π of Γ The set of permutation invariant multipliers is denoted by $\Pi(A_p , A_r)$

In view of Proposition 2.8, the study of $\Pi(A_p , A_r)$ is interesting only in cases (2 9)-(2 11).

The following theorem completely characterizes $\Pi(A_p(\mathbb{T}) , A_r(\mathbb{T}))$, $1 \le r \le 2 < p < \infty$.

Theorem 3.3: Let $1 \le r \le 2 < p < \infty$, then

$$\ell_{\frac{pr}{p-r}}(Z) = \Pi(A_p(\mathbb{T}) , A_r(\mathbb{T}))$$

The proof of the above theorem depends on the following lemma which may be of independent interest:

Lemma 3.4: Let $2 < p < \infty$ and $(a(n)) \in \ell_p(Z)$ be such that $a(n) \ge 0$ and $a(n) = a(-n)$ $\forall$ $n \in \mathbb{N}$. Then there exists a permutation π of Z such that $ao\pi(n) \in (L^1(\mathbb{T}))\hat{}$.

The proof of Lemma 3 4 depends on the following well-known theorem [3]

<u>Theorem 3.5</u>: Let $(a(n))$ be an even, non-negative, decreasing sequence of real numbers such that $\sum_{n=1}^{\infty} \frac{a(n)}{n} < \infty$ Then there exists an $f \in L^1$ such that $\hat{f}(n) = a_n \ \forall\, n \in Z$ and

$$||f||_{L^1} \le C \left(\sum_{n=1}^{\infty} \frac{a_n}{n} \right) + a_0$$

<u>Proof of Lemma 3 4</u>: Let π be a permutation of $\mathbb{N}$ such that $ao\pi$ is decreasing on $\mathbb{N}$ Extend π to Z by defining $\pi(-n) = -\pi(n) \ \forall\, n \in \mathbb{N}$ and $\pi(0)=0$ We show that $ao\pi \in (L^1(\mathbb{T}))\hat{}$ Clearly, $ao\pi \ge 0$, $ao\pi(n) = ao\pi(-n) \ \forall\, n \in \mathbb{N}$ and $ao\pi$ is decreasing to zero on $\mathbb{N}$ Also,

$$\sum_{n=1}^{\infty} \frac{ao\pi(n)}{n} \le \left(\sum_{n=1}^{\infty} (ao\pi(n))^p \right)^{1/p} \left(\sum_{n=1}^{\infty} 1/n^{p'} \right)^{1/p'} < \infty$$

Therefore $ao\pi$ satisfies the conditions of Theorem 3 5 Hence $ao\pi \in (L^1(\mathbb{T}))\hat{}$

This completes the proof of the lemma

<u>Proof of Theorem 3.3</u> It is clear that $\ell_{\frac{pr}{p-r}}(Z) \subseteq \Pi(A_p(\mathbb{T}), A_r(\mathbb{T}))$.

Conversely, suppose $(a(n)) \notin \ell_{\frac{pr}{p-r}}(Z) = M(\ell_p(Z), \ell_r(Z))$, then there exists a sequence $(b(n)) \in \ell_p(Z)$ such that $(a(n)b(n))$ does not belong to $\ell_r(Z)$.

Define

$$c(n) = \max \left(\frac{1}{|n|} + |b(n)|, \frac{1}{|n|} + |b(-n)| \right)$$

Then $(c(n)) \in \ell_p(Z)$, and $(a(n)c(n)) \notin \ell_r(Z)$ Also $(c(n))$ satisfies the conditions of Lemma 3 4, hence there exists a

permutation π of Z such that $(co\pi(n)) \in (L^1(\mathbb{T}))^{\wedge}$. Therefore $(co\pi(n)) \in \hat{A}_p(\mathbb{T})$

It follows that $ao\pi \notin (A_p(\mathbb{T}), A_r(\mathbb{T}))$ as $(ao\pi(n)\, co\pi(n)) \notin \ell_r(Z)$

This completes the proof of the theorem

<u>Remark 3.6</u> Lemma 3 4 can be viewed as an intermediate result between the following

(i)Helgason [9] Let G be a compact abelian group and ϕ a function on Γ Then $\phi \in \ell_2$ if and only if $\phi o\pi \in \hat{L}^1$ for every permutation π of Γ.

(ii) Kahane [12]. There exists a sequence $\phi \in C_o(Z)$ such that $\phi o\pi \notin (L^1(\mathbb{T}))\hat{}$ for any permutation π of Z

The proof of Helgason's result gives the following result about (A_p, A_p)-multipliers

<u>Theorem 3.7</u>: (a) Let $p > 2$ and ϕ a multiplier in $(A_p, A_p) \cap C_o$ Then there exists a permutation π of Γ such that $\phi o\pi$ belongs to the multiplier space (A_p, A_2).

(b) $\Pi(L^1, L^1) \cap C_o = \ell_2$

<u>Proof</u>:(a) Let $E = \{\gamma \in \Gamma |\ \phi(\gamma) \neq 0\}$ Since $\phi \in C_o$, E is countable If E is finite then $\phi o\pi \in (A_p, A_2)$ for every permutation π of Γ. So, we assume without loss of generality that E is infinite Let E_1 be an infinite subset of E such that

$$\sum_{\gamma\in E_1} |\phi(\gamma)|^{2p/(p-2)} < \infty \tag{3 8}$$

Let $E_2 = E\backslash E_1$ .If E_2 is finite then for every permutation π of Γ

$\phi o\pi \in \ell_{\frac{2p}{p-2}} \subseteq (A_p, \ell_2)$ So, we assume without loss of generality

that E_2 is infinite Choose a countably infinite Λ_2 subset F_1 of Γ and define $F_2 = \Gamma \backslash F_1$ Let π be a permutation of Γ mapping F_1 onto E_2 We claim that $\phi o \pi \in (A_p , \ell_2)$. Let $f \in A_p$ .Since $\phi o \pi \in (A_p , A_p)$, choose a $g \in A_p$ such that $\hat{g} = (\phi o \pi) \hat{f}$. Using Holder's inequality and (3 8), we get

$$\sum_{\gamma \in F_2} |\phi o \pi(\gamma)|^2 \, |\hat{f}|^2 < \alpha$$

Hence there exists an $h \in L^2$ such that

$$h = \begin{cases} \phi o \pi(\gamma) \, \hat{f}(\gamma), & \gamma \in F_2 \\ 0, & \text{otherwise} \end{cases}$$

Now $g-h \in L^1_{F_1}$ Since F_1 is a Λ_2 set, we have $g-h \in L^2$ Therefore, $g \in L^2$

(b) The proof in this case follows exactly as in the above case if we choose the set $E_1 \subseteq E$ such that

$$\sum_{\gamma \in E_1} |\phi(\gamma)|^2 < \infty$$

In cases (2 10)-(2.11) we are not able to characterize $\Pi(A_p(\mathbb{T}), A_r(\mathbb{T}))$ However, we prove the following theorem, characterizing a subclass of $\Pi(A_p(\mathbb{T}) . A_r(\mathbb{T}))$

<u>Theorem 3 9</u>· Let $(a(n))$ be a sequence on Z

(i) For $p > 2$, suppose $(a(n) \, \varepsilon(n)) \in \Pi(A_p(\mathbb{T}), A_p(\mathbb{T}))$ for every sequence $(\varepsilon(n))_{n \in Z}$, $\varepsilon(n) = \pm 1$, then $(a(n)) \in \ell_{\frac{2p}{p-2}}(Z)$

(ii)For $2 < r < p < \infty$, suppose $(a(n) \, \varepsilon(n)) \in \Pi(A_p(\mathbb{T}) , A_r(\mathbb{T}))$ for every sequence $(\varepsilon(n))$ mentioned in (i), then $(a(n)) \in \ell_{\frac{2p}{p-2}}(Z)$

<u>Proof</u> (i) Let $(a(n)) \notin \ell_{\frac{2p}{p-2}}(Z) = M(\ell_p(Z),\ell_2(Z))$. Then there exists a sequence $(d(n)) \in \ell_p(Z)$ such that $d(n) \geq 0$, $d(n) = d(-n)$ $\forall n \in \mathbb{N}$, $d(n) \neq 0$ $n \in \mathbb{N}$ and $(a(n)\, d(n)) \notin \ell_2(\mathbb{N})$. Hence by Lemma 3.4, there exists a permutation π of Z such that $(do\pi(n)) \in \hat{A}_p(\mathbb{T})$. Since $(ao\pi(n)\;\; do\pi(n)) \notin \ell_2(Z)$, therefore there exists a sequence $(\varepsilon(n))_{n\in Z}$, $\varepsilon(n) = \pm 1$, such that $(\varepsilon(n)\, ao\pi(n)\, do\pi(n)) \notin (L^1(\mathbb{T}))\hat{}$. Hence $(a(n)\, \varepsilon(n)) \notin \Pi(A_p(\mathbb{T})\,,\, A_p(\mathbb{T}))$, a contradiction.

(ii) The proof of (ii) is similar to that of (i).

This completes the proof of the theorem.

<u>Remark 3.10</u>: Let $P > 2$. Then $\ell_{\frac{2p}{p-2}} \subsetneq \Pi(A_p\,,\, A_p)$ since the constant function 1 on Γ belongs to $\Pi(A_p\,,\, A_p)$.

§ 4

Throughout this section we assume that $1 \leq r \leq 2 < p < \infty$. We have $\ell_{\frac{pr}{p-r}} \subseteq (A_p\,,\, A_r)$, and the proper inclusion is not known. In section 3, we proved that $\ell_{\frac{pr}{p-r}}(Z) = \Pi(A_p(\mathbb{T})\,,\, A_r(\mathbb{T}))$. In this section, we give a sufficient condition on a multiplier $\phi \in (A_p(\mathbb{T})\,,\, A_r(\mathbb{T}))$ so that $\phi \in \ell_{\frac{pr}{p-r}}(Z)$.

It is easy to see that $\phi \in (A_p\,,\, A_r)$ if and only if $|\phi| \in (A_p\,,\, A_r)$. Therefore it is sufficient to characterize non-negative multipliers from A_p to A_r.

<u>Theorem 4 1</u>: Let Φ be a non-negative sequence on Z which is decreasing on both sides of Z Then $\Phi \in (A_p(\mathbb{T}),A_r(\mathbb{T}))$ if and only if $\Phi \in \ell_{\frac{pr}{p-r}}(Z)$

<u>Proof:</u> Clearly, $\Phi \in (A_p(\mathbb{T}) , A_r(\mathbb{T}))$ if and only if

$$\psi(n) = \Phi(n) + \Phi(-n) \in (A_p(\mathbb{T}),A_r(\mathbb{T}))$$

Hence, without loss of generality we may assume that Φ is even and $\Phi(0) = 0$ Let

$$\psi_m(n) = \begin{cases} (\Phi(n))^{r/(p-r)} & \text{on } [-m,m] \\ 0 & \text{otherwise} \end{cases}$$

Then
$$||\Phi\psi_m||_{\ell_r} \le ||\Phi|| \; (||\check{\psi}_m||_{L^1} + ||\psi_m||_{\ell_p}) \tag{4 2}$$

Now by Theorem 3.5, we have

$$||\check{\psi}_m||_{L^1} \le C \sum_{n=1}^{m} \frac{\psi_m(n)}{n}$$

$$\le C \sum_{n=1}^{m} (\frac{1}{n^{p'}})^{1/p'} \quad (\sum_{n=1}^{m} (\Phi(n))^{pr/(p-r)})^{1/p} \tag{4 3}$$

Also $||\Phi\psi_m||_{\ell_r} =(2 \sum_{n=1}^{m} (\Phi(n))^r \; (\Phi(n))^{r^2/(p-r)})^{1/r}$

$$= 2^{1/r}(\sum_{n=1}^{m} (\Phi(n))^{pr/(p-r)})^{1/r} \tag{4 4}$$

Hence combining (4.2)-(4.4), we get

$$2^{1/r}(\sum_{n=1}^{m} (\Phi(n))^{pr/(p-r)})^{1/r} \le ||\Phi|| \; (C' \; (\sum_{n=1}^{m} (\Phi(n))^{pr/(p-r)})^{1/p}$$

$$+ (\sum_{n=1}^{m} \Phi^{pr/(p-r)})^{1/p})$$

It follows that

$$\left(\sum_{n=1}^{m} (\phi(n))^{pr/(p-r)}\right)^{1/r-1/p} \leq ||\phi|| \; (1 + C'),$$

where the right hand side is independent of m Hence $\phi \in \ell_{\frac{pr}{p-r}}(Z)$

This completes the proof of the theorem

<u>Remark 4.5</u>: The proof of Theorem 4 1 actually shows the following Suppose ϕ is a function defined on Z which is decreasing on both sides of Z and $1 \leq r < p < \infty$, $p > 2$ Then $\phi \in (A_p(\mathbb{T}), \ell_r(Z))$ if and only if $\phi \in \ell_{\frac{pr}{p-r}}(Z)$

CHAPTER IV

MULTIPLIERS FROM L_r^p TO ℓ_q

§1. INTRODUCTION

Throughout the chapter, G denotes a compact abelian group, unless stated otherwise We begin with a definition

<u>Definition 1.1</u> Let S be a subset of $L^1(G)$ and $1 \le q < \infty$ A bounded function ϕ on Γ is said to be a multiplier from S to $\ell_q(\Gamma)$ if $\phi\hat{f} \in \ell_q$ for every $f \in S$ The set of multipliers from S to ℓ_q is denoted by (S , ℓ_q)

Suppose $1 \le q \le 2$ Using Plancherel theorem, we see that $\hat{A}_q = \ell_q$. Therefore, in view of Definition 1 1, we have

$$(A_r , A_q) = (A_r , \ell_q), \quad 1 \le r < \infty$$

Next, we introduce some notation Let $\mathcal{F}$ be a subset of $L^1(G)$ By $\mathcal{F}_r$ we denote the set $\{f \in \mathcal{F} \mid \hat{f} \in \ell_r(\Gamma)\}$. Note that $A_r = L_r^1$.

In Chapter III, we discussed (A_r , A_q)-multipliers. We mentioned there that the proper containments in the following inclusions are not known

$$\ell_{\frac{rq}{r-q}} \subseteq (A_r . A_q) , \quad 1 \le q \le 2 < r < \infty \tag{1 2}$$

$$\hat{M}(G) + \ell_{\frac{2r}{r-2}} \subseteq (A_r , A_r) , \quad r > 2 \tag{1.3}$$

$$\hat{M}_{\frac{rq}{r-q}}(G) + \ell_{\frac{2r}{r-2}} \subseteq (A_r , A_q) , 2 < q < r < \infty \tag{1 4}$$

However, (1.2)-(1 4) correspond to a special case (corresponding to p = 1) of the following inclusions

$$\ell_{\frac{rq}{r-q}} \subseteq (L_r^p , A_q), r > 2, 1 \le q < r < \infty \tag{1 5}$$

$$\hat{M}(G) + \ell_{\frac{2r}{r-2}} \subseteq (L_r^p , A_r), r > 2 \tag{1 6}$$

$$\hat{M}_{\frac{rq}{r-q}}(G) + \ell_{\frac{2r}{r-2}} \subseteq (L_r^p , A_q), 2 < q < r < \infty \tag{1 7}$$

We also observe that if $p > 1$, then $(A_r , A_q) \subseteq (L_r^p\ A_q)$ for every r and q In this chapter, we study the proper inclusions in (1.5)-(1 7) for every $p > 1$

The study of (1 5)-(1.7) for $p > 1$ led us to consider (L_r^p , ℓ_q)-multipliers.

In section 2, we study (L^p , ℓ_q)-multipliers. In view of Theorem 2.2 below. the study of (L^p , ℓ_q)-multipliers is interesting only when $1 < p < 2$, $1 \le q < p'$. We have shown that if G is not totally disconnected then

$$(L^{p_1} , \ell_q) \subsetneq (L^{p_2} , \ell_q) , \quad 1 \le p_1 < p_2 \le 2, 1 \le q < p_1' .$$

In section 3, we study (L_r^p , ℓ_q)-multipliers.The main results are.

(i) Let G be a compact abelian group and $p > 1$ Then the inclusions in (1.5)-(1 7) are proper

(ii) Suppose G is not totally disconnected and $r > 2$, $1 \le p < r'$, $1 \le q < r < \infty$. Then

$$(L_r^p , \ell_q) \subsetneq (L^{r'} , \ell_q)$$

In section 4, we study the permutation invariant multipliers from $L_r^p(\mathbb{T})$ to $\ell_q(Z)$

In section 5, we study a sufficient condition on a ϕ belonging to $(L_r^p(\mathbb{T}), \ell_q(Z))$ $(r > 2,\ 1 \le q < r < \infty,\ 1 \le p < r')$ so that $\phi \in \ell_{\frac{rq}{r-q}}(Z)$.

§2. (L^p,ℓ_q)-multipliers

(L^p, ℓ_q)-multipliers have been studied in [11]. We begin by listing some results which follow easily from the following theorem.

Theorem 2 1 ([6] or [11]) : Let G be a compact abelian group and $1 \le q < 2$ Then

$$(C, \ell_q) = \ell_{\frac{2q}{2-q}}$$

Theorem 2.2: Let G be a compact abelian group Then

(a) For $p \ge 2$, $\quad (L^p, \ell_q) = (L^2, \ell_q) = \ell_{\frac{2q}{q-2}}$, $\quad 1 \le q < 2$

$$= \ell_\infty \qquad q \ge 2$$

(b) For $1 < p \le 2$, $\quad (L^p, \ell_q) = \ell_\infty$, $\quad q \ge p'$.

(c) $(L^1, \ell_q) = \ell_q$, $\quad 1 \le q \le \infty$

Proof: We briefly sketch the proofs of (a), (b), and (c)

(a) Clearly $(L^2, \ell_q) \subseteq (L^p, \ell_q) \subseteq (C, \ell_q)$, $p \geq 2$

Now from Theorem 2.1 $(C, \ell_q) = \ell_{\frac{2q}{2-q}}$ for $1 \leq q < 2$

Clearly, $(L^2, \ell_q) = \ell_{\frac{2q}{2-q}}$ for $1 \leq q < 2$

Hence $(L^p, \ell_q) = (L^2, \ell_q) = \ell_{\frac{2q}{2-q}}$ for $1 \leq q < 2$.

Also, clearly $(L^p, \ell_q) = \ell_\infty$ for $q \geq 2$

(b) follows from Hausdorff-Young theorem

(c) Clearly, $\ell_q \subseteq (L^1, \ell_q)$ Let $\phi \in (L^1, \ell_q)$ and let (k_α) be a bounded approximate identity for L^1 such that each k_α is a trigonometric polynomial and $||k_\alpha||_{L^1} \leq 1$, $\forall\ \alpha$ Then

$$||\phi\ \hat{k}_\alpha||_{\ell_q} \leq ||\phi||\ ||k_\alpha||_{L^1} \leq ||\phi||$$

Since $\hat{k}_\alpha \longrightarrow 1$, we conclude that $||\phi||_{\ell_q} \leq ||\phi||$ and (c) follows

This completes the proof of the theorem.

We now study (L^p, ℓ_q)-multipliers when $1 < p < 2$ and $1 \leq q < p'$

Theorem 2.3: Suppose G is not a totally disconnected group and $1 \leq p_1 < p_2 \leq 2$, $1 \leq q < p_1'$ Then

$$(L^{p_1}, \ell_q) \subsetneq (L^{p_2}, \ell_q)$$

First we prove Theorem 2 3 for the case $G = \mathbb{T}$ Then the proof of the general case is reduced to the case $G = \mathbb{T}$, by using the fact that there exists a closed subgroup H of G such that G/H is isomorphic with $\mathbb{T}$.

The proof for $G = \mathbb{T}$ is based on the following known result

<u>Theorem 2.4 ([5])</u>: Let (a_n) be a non-negative, even, decreasing sequence on Z and $1 \le p < \infty$, then there exists an $f \in L^p(\mathbb{T})$ such that $\hat{f}(n) = a_n \; \forall \, n \in Z$ if and only if $\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty$ And in this case

$$C_1 \left(\sum_{n=0}^{\infty} (1+n)^{p-2} a_n^p \right)^{1/p} \le \|f\|_p \le C_2 \left(\sum_{n=0}^{\infty} (1+n)^{p-2} a_n^p \right)^{1/p}$$

for some positive constants C_1 and C_2

<u>Proof of Theorem 2 3</u>: The proof consists of three steps

Step 1 ($G=\mathbb{T}$). First we assume $p_2' > q$. Let $\frac{1}{r} = \frac{1}{q} - \frac{1}{p_2'}$, then by Hausdorff-Young theorem, we get $\ell_r \subseteq (L^{p_2}, \ell_q)$

We show below that there exists a sequence $(a_n) \in \ell_r$ such that $(a_n) \notin (L^{p_1}(\mathbb{T}), \ell_q)$. Let

$$a_n = \begin{cases} \dfrac{1}{|n|^{1/r} \log|n|}, & |n| \ge 2 \\ 0, \text{ otherwise} \end{cases}$$

Then $(a_n) \in \ell_r$ as $r > 1$.

Let

$$b_n = \begin{cases} \dfrac{1}{|n|^{1/p_1'} (\log|n|)^2}, & |n| \ge 2 \\ 0, \quad \text{otherwise} \end{cases}$$

Then $(b\ )$ satisfies the conditions of Theorem 2.4 for $p = p$

Hence $(b_n) \in L^{p_1}$. Now we show that $(a_n b_n) \notin \ell_q$

$$\sum_{-\infty}^{+\infty} a_n^q b_n^q = 2 \sum_{n=2}^{\infty} \frac{1}{n^{q/r} (\log n)^q \, n^{q/p_1'} (\log n)^{2q}}$$

$$= 2 \sum_{n=2}^{\infty} \frac{1}{n^{q(1/r + 1/p_1')} (\log n)^{3q}} = \infty$$

as $\quad q/r + q/p_1' = q\,(1/q - 1/p_2') + q/p_1' = 1 - q(1/p_2' - 1/p_1') < 1$

Hence $(a_n) \notin (L^{p_1}, \ell_q)$

Next, suppose $p_2' \le q$ Then $(L^{p_2}, \ell_q) = \ell_\infty$. Let $a_n = 1 \ \forall\, n \in \mathbb{Z}$, then $a_n b_n = b_n \notin \ell_q$ as $q < p_1'$. Hence$(a_n) \notin (L^{p_1}, \ell_q)$

This completes the proof of step 1

Step 2 Let G be a compact abelian group such that Theorem 2 3 holds for G/H, for some closed subgroup H of G Then it holds for G

<u>Proof</u>: Let $\Phi \in (L^{p_2}(G/H), \ell_q(H^\perp))$ such that $\Phi \notin (L^{p_1}(G/H), \ell_q(H^\perp))$

Let $f \in L^{p_1}(G/H)$ be such that $\Phi\, \hat{f} \notin \ell_q(H^\perp)$ Define $\Phi = 0$ on $\Gamma\backslash H^\perp$

We show that $\Phi \in (L^{p_2}, \ell_q)$ and $\Phi \notin (L^{p_1}, \ell_q)$ Let $g \in L^{p_2}$, then $\Pi_H(g) \in L^{p_2}(G/H)$ and $(\Pi_H(g))^\wedge = \hat{g}$ on $H^\perp$ Hence

$$\Phi\, \hat{g} = \begin{cases} \Phi(\Pi_H(g))^\wedge & \text{on } H^\perp \\ 0 & \text{otherwise} \end{cases}$$

Hence $\Phi\, \hat{g} \in \ell_q(\Gamma)$ as $\Phi(\Pi_H(g))^\wedge \in \ell_q(H^\perp)$. Therefore,$\Phi \in (L^{p_2}, \ell_q)$.

Also, $f\circ\Pi_H \in L^{p_1}(G)$ and $(f\circ\Pi_H)^\wedge = \hat{f}\, \chi_{H^\perp}$. Hence $\Phi(f\circ\Pi_H)^\wedge \notin \ell_q$ and so $\Phi \notin (L^{p_1}, \ell_q)$. This completes the proof of step 2

<u>Step 3</u>: Since G is not totally disconnected, Γ contains an element of infinite order, say, γ_0. Let S denote the subgroup generated by γ_0 and let $H = S^\perp$ Then G/H is isomorphic with the circle group $\mathbb{T}$. Now the proof of the theorem follows from step 1 and step 2.

Let G be a compact abelian group. By Hausdorff-Young theorem and Holder's inequality , we have

$$\ell_{\frac{p'q}{p'-q}} \subseteq (L^p, \ell_q) \ , \ 1 < p < 2 \ , \ q < p' \qquad (2\ 5)$$

In the following theorem we show that the inclusion in (2.5) is proper.

<u>Theorem 2.6</u> : Let G be a compact abelian group and $1<p<2$ Then

$$\ell_{\frac{p'q}{p'-q}} \subsetneq (L^p , \ell_q) \text{ for } q < p'$$

<u>Proof</u>:

Case 1 $1 \le q < 2$ Let E be a Sidon subset of Γ Then

$$\hat{L}^p|_E \subseteq \ell_2(E)$$

Hence, by Holder's inequality, we get

$$\ell_{\frac{2q}{2-q}}(E) \subseteq (L^p , \ell_q)$$

Now choose a sequence $\phi \in \ell_{\frac{2q}{2-q}}(E)$ such that $\phi \notin \ell_{\frac{p'q}{p'-q}}(E)$ Extend ϕ to Γ by putting $\phi = 0$ on $\Gamma\backslash E$ Then $\phi \in (L^p , \ell_q)$, but $\phi \notin \ell_{\frac{p'q}{p'-q}}$.

This completes the proof in case 1.

Case 2. $q \ge 2$. Clearly, $\ell_\infty(E) \subseteq (L^p , \ell_q)$ for any Sidon subset E of Γ. Hence we get that $\ell_{\frac{p'q}{p'-q}} \subsetneq (L^p , \ell_q)$

This completes the proof of the theorem

§ 3. (L^p_r, ℓ_q)-multipliers

In Theorem 3.1 below, we show that the inclusions in (1.5)-(1.7) are all proper when $p > 1$.

<u>**Theorem 3.1**</u> Let G be a compact abelian group and $p > 1$. Then

(a) $\ell_{\frac{rq}{r-q}} \subsetneq (L^p_r, \ell_q)$, $r > 2$, $1 \le q < r < \infty$

(b) $\hat{M}(G) + \ell_{\frac{2r}{r-2}} \subsetneq (L^p_r, A_r)$, $r > 2$

(c) $\hat{M}_{\frac{rq}{r-q}}(G) + \ell_{\frac{2r}{r-2}} \subsetneq (L^p_r, A_q)$, $2 < q < r < \infty$

The proof of part (b) of Theorem 3.1 uses the following result from the theory of Sidon sets.

<u>**Theorem 3.2 ([21])**</u>: Let G be a compact abelian group, E a Sidon subset of Γ and let $\phi \in \ell_\infty(E)$. Then there exists a measure $\mu \in M(G)$ such that $\hat{\mu}|_E = \phi$ and $\hat{\mu}|_{\Gamma\setminus E} \in C_o(\Gamma\setminus E)$ if and only if $\phi \in C_o(E)$.

<u>**Proof of Theorem 3.1**</u>: Let E be a Sidon subset of Γ.

(a) Since $\hat{L}^p|_E \subseteq \ell_2(E)$, $p > 1$, by Holder's inequality, we have

$$\ell_{\frac{2q}{2-q}}(E) \subseteq (L^p, \ell_q) \subseteq (L^p_r, \ell_q), \quad 1 \le q \le 2$$

For $q > 2$, we have $\ell_\infty(E) \subseteq (L^p, \ell_q) \subseteq (L^p_r, \ell_q)$. Also $\ell_{\frac{rq}{r-q}} \subsetneq \ell_{\frac{2q}{2-q}}$, since $r > 2$. Hence (a) follows.

(b) For $r > 2$, we have $\ell_\infty(E) \subseteq (L^p, L^2) \subseteq (L^p_r, A_r)$. Now we show that

$$\ell_\infty(E) \nsubseteq \hat{M}(G) + \ell_{\frac{2r}{r-2}}$$

Let $$\phi_1 = \begin{cases} 1 & \text{on } E \\ 0 & \text{outside } E \end{cases}$$

Then $\phi_1 \in \ell_0(E)$ Now if $\phi_1 = \hat{\mu} + \psi$ for some $\mu \in M(G)$ and $\psi \in \ell_{\frac{2r}{r-2}}$

Then $\hat{\mu} = -\psi$ on $\Gamma\backslash E$, therefore $\hat{\mu} \in C_0(\Gamma\backslash E)$ Then, by Theorem 3 2, applied to $\phi = \hat{\mu}|_E$ we conclude that $\hat{\mu}|_E \in C_0(E)$ Hence $\phi_1|_E \in C_0(E)$, which is false Hence (b) is proved

(c) As $\ell_0(E) \subseteq (L^p, L^2) \subseteq (L^p_r, A_q)$, (c) follows

As mentioned earlier, we have

$$\ell_{\frac{rq}{r-q}} \subseteq (A_r, \ell_q) \subseteq (L^p_r, \ell_q), \quad r > 2,\ 1 \le q < r < \infty,\ p > 1$$

and by Theorem 3 1 $\ell_{\frac{rq}{r-q}} \subsetneq (L^p_r, \ell_q)$ for every $p > 1$ Naturally, one may ask whether (A_r, ℓ_q) is properly contained in (L^p_r, ℓ_q) for every $p > 1$? We show that this is so if $p = r'$ (then $L^{r'}_r = L^{r'}$) In fact, we prove a stronger result:

<u>Theorem 3.3</u>: Suppose G is not totally disconnected Let $r > 2$ and $1 \le p < r'$ Then

$$(L^p_r, \ell_q) \subsetneq (L^{r'}, \ell_q), \quad 1 \le q < r < \infty.$$

We prove Theorem 3 3 for $G = \mathbb{T}$ The proof in the general case can be completed exactly as that of Theorem 2 3

The proof of Theorem 3 3 for $\mathbb{T}$ depends on Lemma 3 5 below We shall also need the following theorem

<u>Theorem 3.4</u>:

(a) [5] (Hardy-Littlewood) If $1 < p \le 2$ and $f \in L^p$, then

$$\{\sum_{n\in Z} |n|^{p-2} |\hat{f}(n)|^p\}^{1/p} \le C_p \|f\|_{L^p}$$

for some constant $C_p > 0$.

(b) [5] Suppose that $1 < p \le q \le p'$ (so that $1 < p \le 2$) Then there exists a constant $C = C_p > 0$ such that

$$\| (|n|)^{((1/p')-(1/q))} \hat{f} \|_{\ell_q} \le C_p \|f\|_{L^p}$$

<u>Lemma 3 5</u>: Let $1 < t \le 2$, and ϕ be a complex-valued function defined on Z, then $\phi \in (L^t, \ell_q)$ if

(a) $$M = \sum_{n \in Z/\{0\}} \frac{|\phi(n)|^{\frac{tq}{t-q}}}{|n|^{\frac{(t-2)q}{t-q}}} < \infty \quad \text{when } q < t \qquad (3\ 6)$$

(b) $M = \sup\limits_{n \in Z} |n|^{1/s} |\phi(n)| < \infty$ when $t \le q \le t'$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{t}$, (3 7)

<u>Proof</u> (a) Let $f \in L^t$, $\alpha = \frac{t}{q}$ (then $\alpha' = \frac{t}{t-q}$) Using Holder's inequality with exponents α and α', we get

$$\sum_{n \in Z/\{0\}} |\phi(n)|^q |\hat{f}(n)|^q = \sum_{n \in Z/\{0\}} |\phi(n)|^q |\hat{f}(n)|^q \, |n|^{\frac{(t-2)q}{t}} \, |n|^{\frac{(2-t)q}{t}}$$

$$\le \Big(\sum_{n \in Z/\{0\}} |\hat{f}(n)|^t |n|^{t-2} \Big)^{q/t} \left(\sum_{n \in Z/\{0\}} \frac{|\phi(n)|^{\frac{tq}{t-q}}}{|n|^{\frac{(t-2)q}{t-q}}} \right)^{\frac{t-q}{t}}$$

$$\le C_t^q \|f\|_{L^t}^q \, M^{\frac{t-q}{t}}$$

Hence $\|(\phi \hat{f})\|_{\ell_q} \le C \|f\|_{L^t}$, for some constant C This proves the theorem in case (a)

(b) Let $f \in L^t$, then

$$\sum_{n \in Z} |\phi(n)|^q |\hat{f}(n)|^q = \sum_{n \in Z} |\phi(n)|^q |\hat{f}(n)|^q \, |n|^{\frac{q}{s}} \, |n|^{-(\frac{q}{s})}$$

$$\le M^q \sum_{n \in Z} |\hat{f}|^q \, |n|^{(\frac{1}{t'} - \frac{1}{q})q}$$

$$= M^q \, \| |n|^{(\frac{1}{t'} - \frac{1}{q})} \hat{f} \|_{\ell_q}^q \le M^q \, C_t^q \, \|f\|_{L^t}^q$$

Hence $\|(\phi\,\hat{f})\|_{\ell_q} \le M\, C_t\, \|f\|_{L^t}$. so that (b) is proved

This completes the proof of the lemma.

Proof of Theorem 3.3. We prove the theorem for $G = \mathbb{T}$

We shall use the following fact if $\psi \in \ell_r$ $(r > 2)$, then

$$\sum_{n\in Z} |n|^{p-2}\, |\psi(n)|^p < \infty \quad \text{when } 1 \le p < r'$$

Indeed, by Holder's inequality, we get

$$\sum_{n\in Z} |n|^{p-2}\, |\psi(n)|^p \le \Big(\sum_{n\in Z} |\psi(n)|^r\Big)^{p/r} \Big(\sum_{n\in Z} |n|^{\frac{(p-2)r}{r-p}}\Big)^{\frac{(r-p)}{r}} < \infty$$

as $\frac{r(2-p)}{r-p} > 1$

Case 1. $q < r'$ We shall construct a ϕ satisfying (3 6) for $t=r'$, and a $\psi \in \ell_r$ such that ψ is non-negative, even, decreasing and $\phi\psi \notin \ell_q$ Then $\phi \in (L^{r'}, \ell_q)$ by Lemma 3 5(a) and $\psi \in \hat{L}^p_r$ for $1 \le p < r'$ by Theorem 2 4. Hence $\phi \notin (L^p_r, \ell_q)$

Define

$$\psi(n) = \begin{cases} \dfrac{1}{|n|^{1/r}(\log|n|)^{1/2}}, & |n| \ge 2 \\ 1, & \text{otherwise} \end{cases}$$

and

$$\phi(n) = \begin{cases} \dfrac{|n|^{\frac{q-r}{qr}}}{(\log|n|)^{(2-q)/2q}}, & |n| \ge 2 \\ 1, & \text{otherwise} \end{cases}$$

Since $r > 2$, ψ satisfies the desired conditions. We show below that ϕ satisfies (3.6) for $t = r'$.

$$\sum_{|n|\geq 2} \frac{|\phi(n)|^{\frac{q\,r'}{r'-q}}}{|n|^{(\frac{r'-2}{r'-q})q}} = 2\sum_{n=2}^{\alpha} \frac{n^{(\frac{q-r}{qr})(\frac{qr'}{r'-q})}}{(\log n)^{(\frac{2-q}{2q})(\frac{qr'}{r'-q})} \quad n^{\frac{q(r'-2)}{r'-q}}}$$

$$= 2\sum_{n=2}^{\alpha} \frac{1}{n\,(\log n)^{\frac{(2-q)r'}{2(r'-q)}}} ,$$

as $(\frac{q-r}{qr})\ (\frac{qr'}{r'-q}) = \left(\frac{(r'-2)q}{r'-q} - 1\right)$ Since $\frac{(2-q)\ r'}{2(r'-q)} > 1$, ϕ satisfies (3 6) Next we show that $\phi\ \psi \notin \ell_q$

$$\sum_{|n|\geq 2} |\phi(n)|^q\ |\psi(n)|^q = 2\sum_{n=2}^{\infty} \frac{n^{\frac{q-r}{r}}}{(\log n)^{\frac{(2-q)}{2}} \quad n^{\frac{q}{r}}\ (\log n)^{\frac{q}{2}}}$$

$$= 2\sum_{n=2}^{\alpha} \frac{1}{n\ \log n} = \alpha$$

Hence the proof of case 1 is complete

Case 2. $r' \leq q$. Define

$$\phi(n) = |n|^{\frac{1}{r}-\frac{1}{q}} \quad \forall\ n \in Z$$

and

$$\psi(n) = \begin{cases} \dfrac{1}{|n|^{1/r}(\log |n|)^{1/q}}, & |n| \geq 2 \\ 1, \text{ otherwise} \end{cases}$$

Then ϕ satisfies (3.7) for $t = r'$ Hence $\phi \in (L^r\ ,\ \ell_q\ )$ Since $q < r$, $\psi \in \ell_r$ Therefore, by Theorem 2.4 $\psi \in \hat{L}^p_r$ for $1 \leq p < r'$ We show that $\phi\ \psi \notin \ell_q$.

$$\sum_{|n|\geq 2} |\phi(n)|^q\ |\psi(n)|^q = 2\sum_{n=2}^{\infty} \frac{n^{(q-r)/r}}{n^{q/r}\ (\log n)} = 2\sum_{n=2}^{\infty} \frac{1}{n\ \log n} = \alpha.$$

Hence $\phi \notin (L^p_r\ ,\ \ell_q\ )$.

This completes the proof of the theorem.

§ 4. Permutation invariant multipliers from L_r^p to ℓ_q

In § 3 of Chapter III we studied the permutation invariant multipliers from A_r to A_q and proved the following results

(A) If $1 \le q \le 2 < r < \infty$, then

$$\Pi(A_r(\mathbb{T}) \ , \ A_q(\mathbb{T})) = \ell_{\frac{rq}{r-q}}(Z)$$

(B) Let $(a(n))$ be a function defined on Z Then, if $r > 2$ and $(a(n)\cdot\varepsilon(n)) \in \Pi(A_r(\mathbb{T}), A_r(\mathbb{T}))$ for every sequence $(\varepsilon(n))$ with range in $\{\pm 1\}$, then $(a(n)) \in \ell_{\frac{2r}{r-2}}(Z)$

In this section, we prove the following related results

<u>Theorem 4 1</u>· Suppose $r > 2$, $1 \le q < r < \infty$ and $1 \le p < r'$. Then

$$\Pi(L_r^P(\mathbb{T}) \ , \ \ell_q(Z)) = \ell_{\frac{rq}{r-q}}(Z)$$

<u>Theorem 4.2</u>: Suppose $r > 2$ and $1 \le p < r'$ If $(a(n))$ is a sequence such that $(a(n)\ \varepsilon(n)) \in \Pi(L_r^P(\mathbb{T}) \ , \ A_r(Z))$ for every sequence $(\varepsilon(n))$ with range in $\{\pm 1\}$, then $(a(n)) \in \ell_{\frac{2r}{r-2}}(Z)$

The proofs of Theorems 4 1 and Theorem 4 2 are similar to the proofs of Theorem 3.3 and 3 9 , respectively, of Chapter III. Recall that the proofs of Theorem 3 3 and 3.9 of Chapter III depend on Lemma 3.4 there. We give below the analogue of Lemma 3.4 of Chapter III

<u>Lemma 4.3</u>: Suppose $r > 2$ and $1 \le p < r'$ If $(a(n)) \in \ell_r(Z)$ is such that $a(n) \ge 0$ and $a(n) = a(-n)$, $\forall\, n \in N$, then there exists a permutation π of Z such that $a \circ \pi \in (L^p(\mathbb{T}))\hat{}$

Using Theorem 2 4, this can be proved as Lemma 3 4 of Chapter III

<u>Theorem 4 4</u>. Let $r > 2$ and $1 \le q < r < \infty$ Then

$$\Pi(L^p_r(\mathbb{T})\ ,\ \ell_q(Z)) \subsetneq \Pi(L^{r'}(\mathbb{T})\ ,\ \ell_q(Z)),\ 1 \le p < r'$$

Theorem 4 4 can be proved as Theorem 3 3 by using the following analogue of Lemma 3 5.

<u>Lemma 4.5</u>. Let $1 < t \le 2$ and ϕ be a complex-valued function defined on Z Suppose either of the following conditions hold

(a) $q < t$ and

$$M = \sum_{n \in Z} \frac{|\phi(n)|^{\frac{tq}{t-q}}}{(1+|\pi(n)|)^{\frac{(t-2)q}{t-q}}} < \infty$$

for some permutation π of Z

(b) $t < q \le t'$, $\frac{1}{s} = \frac{1}{q} - \frac{1}{t}$, and

$$M = \sup_{n \in Z} |\pi(n)|^{1/s}\ |\phi(n)| < \infty$$

for some permutation π of Z

Then $\phi \in (L^t\ ,\ \ell_q)$.

In the following corollary of Theorem 4.4, we show that Theorem 4 1 is false if $p \ge r'$

<u>Corollary 4.6</u>: Suppose $r > 2$, $1 \le q < r < \infty$ and $p \ge r'$ Then

$$\ell_{\frac{rq}{r-q}}(Z) \subsetneq \Pi(L^p_r(\mathbb{T})\ ,\ \ell_q(Z))$$

Proof Since $p \geq r'$, $L^p_r(\mathbb{T}) = L^p(\mathbb{T}) \subseteq L^{r'}(\mathbb{T})$ Hence

$$\Pi(L^{r'}(\mathbb{T}), \ell_q(Z)) \subseteq \Pi(L^p_r(\mathbb{T}), \ell_q(Z))$$

Also

$$\ell_{\frac{rq}{r-q}}(Z) \subseteq \Pi(L^1_r(\mathbb{T}), \ell_q(Z))$$

Now by Theorem 4 4, we have

$$\Pi(L^1_r(\mathbb{T}), \ell_q(Z) \subsetneq \Pi(L^{r'}(\mathbb{T}), \ell_q(Z)$$

This proves the corollary

§ 5

In § 4 of Chapter III, we proved that if ϕ is a non-negative multiplier from $A_r(\mathbb{T})$ to $\ell_q(Z)$ $(1 \leq q < r < \infty,\ r > 2)$ which is monotonically decreasing on both sides of Z then $\phi \in \ell_{\frac{rq}{r-q}}(Z)$. In this section, we show that the result still holds if $A_r(\mathbb{T})$ is replaced by $L^p_r(\mathbb{T})$, $1 \leq p < r'$ The proof of the following theorem is similar to that of Theorem 4 1 in Chapter III.

Theorem 5.1: Let $r > 2$, $1 \leq p < r'$, $1 \leq q < r < \infty$ and ϕ be a non-negative function defined on Z which is monotonically decreasing on both sides of Z. Then $\phi \in (L^p_r(\mathbb{T}), \ell_q(Z))$ if and only if $\phi \in \ell_{\frac{rq}{r-q}}(Z)$.

Proof: Clearly $\phi \in (L^p_r, \ell_q)$ if and only if $\psi \in (L^p_r, \ell_q)$, where $\psi(n)=\phi(n)+\phi(-n)$. Hence without loss of generality, we may assume

that ϕ is even and $\phi(0) = 0$. Let

$$\psi_m(n) = \begin{cases} (\phi(n))^{\frac{q}{r-q}}, & -m \le n \le m \\ 0, & \text{otherwise} \end{cases}$$

Then

$$||\phi\,\psi_m||_{\ell_q} \le ||\phi||(||\check{\psi}_m||_{L^p} + ||\psi_m||_{\ell_r}) \tag{5 2}$$

Now by Theorem 2 4,

$$||\check{\psi}_m||_{L^p} \le C\,(\sum_{n=1}^{m} n^{(p-2)}\,(\psi_m(n))^p)^{1/p}$$

$$\le C\left(\sum_{n=1}^{m}(\psi_m(n))^r)^{p/r}\,(\sum_{n=1}^{m} n^{(p-2)(\frac{r}{r-p})})^{\frac{r-p}{p}}\right)^{1/p}$$

$$\le C_1\,(\sum_{n=1}^{m}\,(\phi(m))^{\frac{rq}{r-q}})^{1/r} \tag{5 3}$$

since $\frac{(2-p)r}{r-p} > 1$.

Also $||\phi\,\psi_m||_{\ell_q} = (2\sum_{n=1}^{m}(\phi(n))^q\,(\phi(n))^{\frac{q^2}{r-q}})^{1/q}$

$$= (2\sum_{n=1}^{m}\,(\phi(n))^{\frac{rq}{r-q}})^{1/q} \tag{5.4}$$

and $||\psi_m||_{\ell_r} = (2\sum_{n=1}^{m}\,(\phi(n))^{\frac{rq}{r-q}})^{1/r}$ (5.5)

Substituting (5.3)-(5.5) in (5.2), we get

$$2^{1/q}(\sum_{n=1}^{m}\,(\phi(n))^{\frac{rq}{r-q}})^{(1/q)-(1/r)} \le ||\phi||\,(1 + C_1)$$

Hence $||\phi||_{\ell_{\frac{rq}{r-q}}} \le C_2\,||\phi||$. This completes the proof of the theorem

<u>Remark 5.6</u>: Theorem 5.1 is false if $p = r'$. To see this, consider the sequence ϕ constructed in the proof of Theorem 3 3

CHAPTER V

THE CONJUGATE OPERATOR

§ 1. INTRODUCTION

We begin with some definitions

<u>Ordered groups</u> Suppose P is a semigroup in a locally compact abelian group G which is closed and has two additional properties

$$P \cap (-P) = \{0\}, \quad P \cup (-P) = G \tag{1 1}$$

Then P induces an order on G For $x, y \in G$, we say that $x \geq y$ if $x - y \in P$ Then $\geq$ is transitive and (1 1) shows that each pair x, y satisfies one and only one of the relations $x > y$, $x = y$, $y > x$. (G,P) is called an ordered group.

It is known [26] that if G is a compact abelian group then Γ can be ordered if and only if G is connected

<u>Conjugate functions</u>: Let G be compact and connected so that Γ can be ordered. With respect to any fixed order P on Γ, we define the conjugate operator as follows Let f be a trigonometric polynomial on G, then the conjugate of f is $\tilde{f} = \sum_{\gamma\in\Gamma} -i \ \mathrm{sgn}_P(\gamma) \ \hat{f}(\gamma)\gamma$, where

$$\mathrm{sgn}_p(\gamma) = \begin{cases} 1 & \gamma \in P\setminus\{0\} \\ 0, & \gamma = 0 \\ -1, & \text{otherwise} \end{cases}$$

The operator $f \longrightarrow \tilde{f}$ so defined on the space of trigonometric

polynomials is called the conjugate operator ($\tilde{f}$ is also called the Hilbert transform of f)

In this chapter, we study the conjugate operator on $A_q(G)$-spaces for a compact, connected abelian group G. Using Plancherel theorem we see that if $1 \le q \le 2$ then $A_q(G) \subseteq L^2(G)$ and the conjugate operator is bounded from $(\mathcal{T}, \|\ \|_{A_q})$ to itself Hence the study of the conjugate operator on $A_q(G)$-spaces is interesting only when $q > 2$ It is easy to see that the conjugate operator is bounded from $(\mathcal{T}, \|\ \|_{A_q})$ to itself if and only if it is bounded from $(\mathcal{T}, \|\ \|_{A_q})$ to $(\mathcal{T}, \|\ \|_{L^1})$

In section 3, we prove that if $q > 2$ and Φ is any Young's function, then the conjugate operator is not bounded from $(\mathcal{T}, \|\ \|_{A_q})$ to $(\mathcal{T}, \|\ \|_{L^\Phi})$ We prove this result using Rudin-Shapiro polynomials on a nondiscrete locally compact abelian group as given in [11]. For the sake of completeness, we give their construction in section 2

The proof of the above mentioned result shows that if G is an infinite compact abelian group, $q > 2$ and Φ is a Young's function then $A_q \nsubseteq L^\Phi$ In section 4, we show that this is true for any nondiscrete locally compact abelian group

§ 2 Rudin-Shapiro Polynomials

Let G be a nondiscrete locally compact abelian group and Γ its dual group. Let $u \in L^1(G)$ be such that $\hat{u} \in C_c(\Gamma)$ Let $E_o = \mathrm{supp}(\hat{u})$ and $f_o = g_o = u$ Choose $\gamma_o \notin E_o - E_o$ and define

$f_1 = f_o + \gamma_o g_o$

$g_1 = f_o - \gamma_o g_o$

and $E_1 = E_o \cup (\gamma_o + E_o) = \mathrm{supp}(\hat{f}_1) = \mathrm{supp}(\hat{g}_1)$

Clearly E_1 is compact Next choose $\gamma_1 \notin E_1 - E_1$ and define

$f_2 = f_1 + \gamma_1 g_1$

$g_2 = f_1 - \gamma_1 g_1$

and $E_2 = E_1 \cup (\gamma_1 + E_1) = \mathrm{supp}(\hat{f}_2) = \mathrm{supp}(\hat{g}_2)$.

Then construct sequences (f_n), (g_n), (E_n) and (γ_n) inductively such that $\gamma_n \notin E_n - E_n$ and

$f_{n+1} = f_n + \gamma_n g_n$

$g_{n+1} = f_n - \gamma_n g_n$

$E_{n+1} = E_n \cup (\gamma_n + E_n) = \mathrm{supp}(\hat{f}_{n+1}) = \mathrm{supp}(\hat{g}_{n+1})$

Then we have

$$m(E_{n+1}) = 2\, m(E_n) = 2^{n+1}\, m(E_o) \qquad (2\ 1)$$

And
$$(|f_n|^2 + |g_n|^2) = (f_{n-1} + \gamma_{n-1} g_{n-1})(\bar{f}_{n-1} + \bar{\gamma}_{n-1} \bar{g}_{n-1}) + (f_{n-1} - \gamma_{n-1} g_{n-1})(\bar{f}_{n-1} - \bar{\gamma}_{n-1} \bar{g}_{n-1})$$

$$= 2\,(|f_{n-1}|^2 + |g_{n-1}|^2) = 2^{n+1}\, |u|^2, \quad \forall\, n \geq 0 \qquad (2.2)$$

Also $\hat{f}_n = \hat{f}_{n-1} + (\gamma_{n-1}\, g_{n-1})^{\wedge} = \hat{f}_{n-1} + \tau_{\gamma_{n-1}}\, \hat{g}_{n-1}$.

Hence, by the choice of γ_n, we see that

$$\mathrm{supp}(\hat{f}_{n-1}) \cap \mathrm{supp}(\tau_{\gamma_{n-1}} \hat{g}_{n-1}) = \emptyset$$

From (2 1) we get

$$||\hat{f}_n||_{\ell_r} = ||\hat{g}_n||_{\ell_r} = 2^{n/r} \, ||\hat{u}||_{\ell_r} \qquad \forall\, n \geq 0, \ \forall\, r \geq 1 \quad (2\ 3)$$

In subsequent sections, we need estimates for $||f_n||_{L^\Phi}$, $||g_n||_{L^\Phi}$ where Φ is a Young's function We estimate these below

Using (2 2) we get

$$|f_n| \ (|g_n|) \leq 2^{(n+1)/2} \, |u|$$

Hence $$||f_n||_{L^\Phi} \ (||g_n||_{L^\Phi}) \leq 2^{(n+1)/2} \, ||u||_{L^\Phi} \qquad (2\ 4)$$

On the other hand $$(|f_n| + |g_n|)^2 \geq |f_n|^2 + |g_n|^2 = 2^{n+1} \, |u|^2$$

Hence $$|f_n| + |g_n| \geq 2^{(n+1)/2} \, |u| \qquad (2\ 5)$$

Now, let

$$h_n = \begin{cases} f_n & \text{if } ||f_n||_{L^\Phi} \geq ||g_n||_{L^\Phi} \\ g_n & \text{otherwise} \end{cases}$$

Then (2.5) gives us $$||h_n||_{L^\Phi} \geq 2^{(n-1)/2} \, ||u||_{L^\Phi} \qquad (2.6)$$

<u>Remark 2.7</u>: If G is a compact and connected group, P a fixed order on Γ and $\mathrm{supp}(\hat{u}) \subseteq P$, then we can choose (γ_n) in the above construction so that $\mathrm{supp}(\hat{f}_n) = \mathrm{supp}(\hat{g}_n) \subseteq P$, $\forall\, n \geq 0$

§ 3. Conjugation in A_q-spaces

We first state some known results

For the following theorem we refer to [1] or [26]

<u>Theorem 3.1</u>: (a) Let G be a compact and connected abelian group Let P be any fixed order on Γ Then for each $q > 1$ there exists a constant C_q, depending only on q such that

$$||\tilde{f}||_{L^q} \le C_q \; ||f||_{L^q} \quad \forall \; f \in \mathcal{T}$$

Hence the conjugate operator can be extended to a bounded linear operator on $L^q(G)$.

(b) Let

$$\phi(t) = \begin{cases} \dfrac{t^2}{e}, & 0 \le t \le e \\ t(\log t), & t > e \end{cases}$$

Then $$||\tilde{f}||_{L^1} \le C \; ||f||_{L^\phi} \quad \forall \; f \in \mathcal{T},$$

where C is a constant independent of f .

<u>Remark 3.2</u>: (i) Part (a) in Theorem 3.1 is false for $q = 1$ [13]

(ii) A partial converse to Theorem 3.1(b) is known for the circle group $\mathbb{T}$ [34]. Let $f \ge 0$, $f \in L^1(\mathbb{T})$ be such that $\tilde{f} \in L^1(\mathbb{T})$ then $f \in L \log^+ L \equiv L^\phi$.

We explore the analogue of part (b) of Theorem 3.1 where $L \log^+ L$ is replaced the by space A_q :Does there exist a constant $C_q > 0$ satisfying

$$||\tilde{f}||_{L^1} \le C_q \; ||f||_{A_q} \quad \forall \; f \in \mathcal{T} \tag{3.3}$$

For $1 \le q \le 2$, (3 3) holds with $C_q = 1$ Indeed,

$$\|\tilde{f}\|_{L^1} \le \|\tilde{f}\|_{L^2} \le \|f\|_{L^2} \le \|f\|_{A_q} \qquad \forall\ f \in \mathcal{T}$$

For $q > 2$, we do not know whether there exists a constant C_q satisfying (3 3) However, we prove below that the conjugate operator is not bounded from $(\mathcal{T}, \|\ \|_{A_q})$ to $(\mathcal{T}, \|\ \|_{L^\Phi})$ for any Young's function Φ

<u>Theorem 3 4</u>. Let G be a compact, connected abelian group and P any fixed order on Γ Suppose Φ is a Young's function and $q > 2$, then the conjugate operator from $(\mathcal{T}, \|\ \|_{A_q})$ into $(\mathcal{T}, \|.\|_{L^\Phi})$ is unbounded

<u>Proof</u>. Consider the operator $S : \mathcal{T} \longrightarrow \mathcal{T}$ defined by

$$S(f) = \sum_{\gamma \ge 0} \hat{f}(\gamma)\ \gamma$$

Then we see that

$$\tilde{f} = -i\ (2\ S(f) - f - \hat{f}(0)) \quad \forall\ f \in \mathcal{T}$$

Suppose there exists a constant $C > 0$ such that

$$\|\tilde{f}\|_{L^\Phi} \le C\ \|f\|_{A_q} \quad \forall\ f \in \mathcal{T}$$

Then it is easy to see that there exists a constant $B>0$ such that

$$\|S(f)\|_{L^\Phi} \le B\|f\|_{A_q} \quad \forall\ f \in \mathcal{T} \qquad (3.5)$$

We show below that (3.5) is false for every Young's function Φ if $q > 2$.

Since Φ is a Young's function, we get $L^\Phi \subsetneq L^1$. Therefore there exists a sequence $(v_n) \subseteq \mathcal{T}$ such that

$$\|v_n\|_{L^1} \longrightarrow 0, \text{ and } \|v_n\|_{L^\Phi} \longrightarrow \infty \text{ as } n \longrightarrow \infty. \qquad (3.6)$$

Without loss of generality, we may assume that $\mathrm{supp}(\hat{v}_n) \subseteq P$ (If not, choose the smallest element in $\mathrm{supp}(\hat{v}_n)$, say γ_n, and replace v_n by $\gamma_n v_n$ Then (3.6) holds for $\gamma_n v_n$, and $\mathrm{supp}((\gamma_n v_n)\hat{}) \subseteq P$)
Let m_n be an increasing sequence of natural numbers such that

$$|\mathrm{supp}(\hat{v}_n)| \le m_n ,$$

Let

$$u_n = \frac{v_n}{2^{m_n/2}}$$

For each u_n, construct a sequence $(h_m^{(n)})$, as in section 2 with $\mathrm{supp}(\hat{h}_m^{(n)}) \subseteq P$ Then, from (2 3), (2 4) and (2 6) of § 2, we get

$$||\hat{h}_m^{(n)}||_{\ell_q} = 2^{m/q}\ ||\hat{u}_n||_{\ell_q} \tag{3 7}$$

$$||h_m^{(n)}||_{L^1} \le 2^{(m+1)/2}\ ||u_n||_{L^1} \tag{3 8}$$

$$||h_m^{(n)}||_{L^\Phi} \ge 2^{(m-1)/2}\ ||u_n||_{L^\Phi} \tag{3 9}$$

Let $h_n = h_{m_n}^{(n)'}$. Using (3.6)-(3 8), we get

$$||\hat{h}_n||_{\ell_q} = 2^{m_n(\frac{1}{q} - \frac{1}{2})}||\hat{v}_n||_{\ell_q}$$

$$\le 2^{m_n(\frac{1}{q} - \frac{1}{2})} m_n^{1/q}\ ||v_n||_{L^1} \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{3.10}$$

Further

$$||h_n||_{L^1} \le 2^{1/2}\ ||v_n||_{L^1} \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{3.11}$$

and

$$||h_n||_{L^\Phi} \ge 1/(2)^{1/2}\ ||v_n||_{L^\Phi} \longrightarrow \infty \text{ as } n \longrightarrow \infty \tag{3 12}$$

Hence from (3.10)-(3 12) we see that (3 5) is false.

This completes the proof of the theorem

Remark 3 13: Let G be an infinite compact abelian group

(i) The proof of Theorem 3.4 given above shows that if $q > 2$ then $A_q \nsubseteq L^{\phi}$, for any Young's function ϕ.

(ii) Part (i) shows that if $q > 2$ then there exists an $f \in A_q$ such that $f \notin L \log L$. If we could construct such a function which, in addition, is non-negative then by Remark 3 2 (ii), it would follow that $\tilde{f} \notin L^1(\mathbb{T})$ And then we would be able to conclude that the conjugate operator is unbounded from $A_q(\mathbb{T})$ to $L^1(\mathbb{T})$ However, we have not been able to resolve this problem

§ 4

In Remark 3.13(i), we observed that if G is an infinite compact abelian group and ϕ is a Young's function then $A_q \nsubseteq L^{\phi}(G)$ for any $q > 2$. We now show that the same result holds for any nondiscrete locally compact abelian group G In fact, we prove a more general result

Theorem 4.1: Let G be a nondiscrete locally compact abelian group and ϕ a Young's function. Let μ be an unbounded Radon measure on Γ Then for $q > 2$ we have

$$A_q(G,\mu) \nsubseteq L^{\phi}(G)$$

where $A_q(G,\mu) = \{f \in L^1(G) : \hat{f} \in L^q(\Gamma,\mu)\}$

Proof: Since $L^1 \nsubseteq L^{\phi}$, we get a sequence $(v_n) \subseteq L^1$ such that $(\hat{v}_n) \subseteq C_c(\Gamma)$,

$$||v_n||_{L^1} \longrightarrow 0, \text{ and } ||v_n||_{L^{\phi}} \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Let (m_n) be an increasing sequence of natural numbers satisfying

$$\mu(\operatorname{supp} \hat{v}_n) \le m_n$$

Let
$$u_n = \frac{v_n}{2^{m_n}}$$

Now by repeating the argument of the proof of Theorem 3.4, we get a sequence $(h_n) \subseteq L^1$ such that $(\hat{h}_n) \subseteq C_c(\Gamma)$ and

$$||h_n||_{A_q(G,\mu)} \longrightarrow 0$$

$$||h_n||_{L^\Phi} \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

Hence $\quad A_q(G,\mu) \nsubseteq L^\Phi$

This completes the proof of the theorem.

Corollary 4.2. Let G be a nondiscrete abelian group If $q > 2$ then $A_q \nsubseteq L^p$ for every $p > 1$

Remark 4 3. (a) H C Lai [15] proved that for a nondiscrete locally compact abelian group G and $1 < p < 2$, $L^1 \cap L^p \subsetneq A_p$, It follows from the Hausdorff Young theorem that $L^1 \cap L^p \subseteq A_p$, The proper inclusion then follows from Corollary 4.2

(b) Let G be a nondiscrete compact abelian group. Tewari and Gupta [29] proved that

$$A_p(G) \subsetneq A_q(G), \quad 1 \le p < q < \infty \tag{4 4}$$

Later on, Tewari and Parthasarthy [31] generalized (4.5) to

$$A_p(G,\mu) \subsetneq A_q(G,\mu), \quad 1 \le p < q < \infty, \tag{4.5}$$

when μ is an unbounded Radon measure on Γ.

The method of the proof of Theorem 4.1 can be used to prove (4.5) when $1 \le p < q \le 2$ or when $1 \le p < 2 < q$.

CHAPTER VI

THE CLASS $L(LOG\ L)^{\alpha}$ AND SOME LACUNARY SETS

§ 1. INTRODUCTION

In [33] Zygmund proved that if f is a function on the circle group $\mathbb{T}$ such that $|f|\ (\log^{+}|f|)^{1/2} \in L^{1}(\mathbb{T})$ and E is a Hadamard set of positive integers then $\sum_{n\in E} |\hat{f}(n)|^{2} < \infty$ Hewitt and Ross ([11], p 446) pointed out that this phenomenon has not been explored for Sidon sets and groups other than $\mathbb{T}$ In this chapter, we investigate this and prove the following generalization of Zygmund's result

Let G be a compact abelian group and let Γ be its dual group Let

$$B_{\alpha} = \{f : |f|(\log^{+}|f|)^{\alpha} \in L^{1}(G)\}, \qquad \alpha > 0$$

If E is a Sidon subset of Γ and $0<\alpha\leq 1/2$ then $\hat{B}_{\alpha}|_{E} \subseteq \ell_{1/\alpha}(E)$ and there exists a Sidon subset E of Γ such that $\hat{B}_{\alpha}|_{E} \nsubseteq \ell_{r}(E)$ for $r < 1/\alpha$ (corollary 3.3).

We then use this result to derive some results about multiplier spaces of certain subspaces of L^{1}

We investigated the existence of non-Sidon sets E for which $\hat{B}_{\alpha}|_{E} \subseteq \ell_{1/\alpha}(E)$. If E is a Sidon set, then E + E is a non Sidon set For particular Sidon subsets of Z $(E = \{2^{k}\}_{k\in\mathbb{N}})$ we found

that if $E_k = E + E + \ldots + E$ (k-times) then $\hat{B}_\alpha|_{E_k} \subseteq \ell_{k/\alpha}(E_k)$ for $0 < \alpha \le k/2$ and $\hat{B}_\alpha|_{E_k} \nsubseteq \ell_r(E_k)$, $r < k/\alpha$ These considerations led us to define and study a new class of lacunary sets, called $\Lambda_{\alpha\ \beta}$ sets .

<u>Definition</u> A subset $E \subseteq \Gamma$ is called a $\Lambda_{\alpha,\beta}$ set if

$$\hat{B}_\alpha|_E \subseteq \ell_{2\beta/\alpha}(E), \ 0 < \alpha \le \beta$$

In view of this definition, the above mentioned result of Zygmund states that a Hadamard set of positive integers is a $\Lambda_{1/2,1/2}$ set and our generalization of the result states that a Sidon subset of Γ is a $\Lambda_{\alpha,1/2}$ set, where $0 < \alpha \le 1/2$.

In section 2, we give a characterization of $\Lambda_{\alpha,\beta}$ sets in Theorem 2 6 This is the main result of this section As a corollary, using a result of Pisier [23], we get a characterization of Sidon-sets As another consequence we get $\Lambda_{\alpha_1,\beta} \subseteq \Lambda_{\alpha_2,\beta}$, $0 < \alpha_2 < \alpha_1 \le \beta$ We also use Theorem 2 6 to give some examples of $\Lambda_{\alpha,\beta}$ sets. We have also included some applications to certain multiplier problems.

In section 3, we provide a partial answer to the problem of deciding whether the class of $\Lambda_{\alpha,\beta}$ sets are distinct for distinct indices β. We prove that for each $k \in \mathbb{N}$, there exists a subset $E \subseteq \Gamma$ which is a $\Lambda_{\alpha,k/2}$ set for $0 < \alpha \le k/2$, but not a $\Lambda_{\alpha,\beta}$ set for $0 < \alpha \le \beta < k/2$. This is a consequence of Theorem 3.1, whose proof takes up all of section 3 We have not been able to prove the distinctness of $\Lambda_{\alpha,\beta}$ sets in the index α.

§ 2

In this section we prove the main theorem of this chapter giving a characterization of $\Lambda_{\alpha,\beta}$ sets. First we prove an analogue of the following known theorem about Sidon subsets of Γ.

Theorem 2.1 ([21]): Let E be a subset of Γ. Then the following statements are equivalent

(a) E is a Sidon set

(b) $A_E^p = \{f \in C : \hat{f} \in \ell_p(E)\} \subseteq A(G)$ for some $p > 1$

(c) $A_E^{1+} = \bigcap_{1<p<\infty} A_E^p \subseteq A(G)$

(d) $(A(\omega))_E = \{f \in C : \omega \hat{f} \in \ell_1(E)\} \subseteq A(G)$ for some $\omega \in C_o$

Now we state and prove an analogue of Theorem 2.1 for $\Lambda_{\alpha,\beta}$ sets.

Theorem 2.2. Let E be a subset of Γ. Then the following statements are equivalent.

(i) E is a $\Lambda_{\alpha,\beta}$ set

(ii) $\hat{B}_{\alpha,r}|_E \subseteq \ell_{\frac{2\beta}{\alpha}}(E)$ for some $r > 2\beta/\alpha$

(iii) $(\bigcap_{r>2\beta/\alpha} B_{\alpha,r})^{\wedge} \subseteq \ell_{\frac{2\beta}{\alpha}}(E)$.

(iv) $((B_\alpha)(\omega))^{\wedge}|_E = \{f \in B_\alpha \mid \omega \hat{f} \in \ell_{\frac{2\beta}{\alpha}}(E)\} \subseteq \ell_{\frac{2\beta}{\alpha}}(E)$ for some $\omega \in C_o$

Proof: It is clear from the definition of $\Lambda_{\alpha,\beta}$ sets that (i) implies (ii)-(iv) $\forall\, r > 2\beta/\alpha$ and for each $\omega \in C_o$. For the reverse implications, we suppose that E is not a $\Lambda_{\alpha,\beta}$ set. We shall prove that each of the statements (ii)-(iv) fails. Let F be a finite subset of E, then E/F is not a $\Lambda_{\alpha,\beta}$ set. Therefore

$$\sup\left\{ \frac{\|\hat{f}|_{E/F}\|_{\ell_{2\beta/\alpha}}}{\|f\|_{B_\alpha}} : f \in \mathcal{T},\ f \neq 0\right\} = \infty \qquad (2.3)$$

Let $\omega \in C_o$ Then by (2 3) there exists a sequence $(f_n)_{n=1}^{\infty} \subseteq \mathcal{T}$ satisfying

(a) $||\hat{f}_n|_E||_{\ell_{2\beta/\alpha}} = \dfrac{1}{n^{\alpha/2\beta}}$

(b) $||f_n||_{B_\alpha} \le 2^{-n}$

(c) $\operatorname{supp}(\hat{f}_1) \cap E \subseteq E \setminus \{\gamma \in E|\ |\omega(\gamma)| \ge 1\}$ and for $n \ge 2$

$\operatorname{supp}(\hat{f}_n) \cap E \subseteq E \setminus [\bigcup_{k=1}^{n-1} ((\operatorname{supp} \hat{f}_k \cap E) \cup \{\gamma \in E|\ |\omega(\gamma)| \ge 1/n\})]$

Let $f = \sum_{n=1}^{\infty} f_n$ Then

$$\sum_{n=1}^{\infty} ||f_n||_{B_\alpha} \le \sum_{n=1}^{\infty} 2^{-n} < \infty \quad .$$

Therefore $f \in B_\alpha$ Further, the sets $\operatorname{supp}(\hat{f}_n) \cap E$ are pairwise disjoint Hence for each $r > 2\beta/\alpha$, we have

$$||\hat{f}|_E||_{\ell_r}^r = \sum_{n=1}^{\infty} ||\hat{f}_n|_E||_{\ell_r}^r \le \sum_{n=1}^{\infty} ||\hat{f}_n|_E||_{\ell_{2\beta/\alpha}}^r$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\alpha r/(2\beta)}} < \infty$$

Therefore $f \in B_{\alpha,r}$ $\forall\ r > 2\beta/\alpha$. However,

$$||\hat{f}|_E||_{\ell_{2\beta/\alpha}}^{2\beta/\alpha} = \sum_{n=1}^{\infty} ||\hat{f}_n|_E||_{\ell_{2\beta/\alpha}}^{2\beta/\alpha} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Therefore (iii) fails and consequently so does (ii).

Also $$||\omega\, \hat{f}|_E||_{\ell_{2\beta/\alpha}}^{2\beta/\alpha} = \sum_{n=1}^{\infty} ||\omega\, \hat{f}_n|_E||_{\ell_{2\beta/\alpha}}^{2\beta/\alpha}$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n^{2\beta/\alpha}} ||\hat{f}_n|_E||_{\ell_{2\beta/\alpha}}^{2\beta/\alpha}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{1+(2\beta/\alpha)}} < \infty \ .$$

Hence we have shown that (ii)-(iv) fail. This completes the proof of the theorem.

Now we prove two simple lemmas Lemma 2.4 below will be needed in the proof of Theorem 2 6, and we use Lemma 2 5 to construct examples of $\Lambda_{\alpha,\beta}$ sets

<u>Lemma 2.4</u> Let E be a subset of Γ, $\alpha > 0$ and $1 < r \le \alpha$ Then $\hat{B}_\alpha|_E \subseteq \ell_r(E)$ if and only if $\ell_{r'}(E) \subseteq (B_\alpha^{*}\hat{)}|_E$, where $1/r + 1/r' = 1$.

<u>Proof</u>. If $r = \infty$, the result is obvious, so assume $r < \infty$ Suppose $\hat{B}_\alpha|_E \subseteq \ell_r(E)$ Then by the closed graph theorem there exists a constant $C > 0$ such that

$$||\hat{f}|_E||_{\ell_r(E)} \le C\ ||f||_{B_\alpha} \qquad \forall\ f \in B_\alpha$$

Let $\phi \in \ell_{r'}(E)$, we define a linear functional on B_α by

$$K_\phi(f) = \sum_{\gamma\in E} \phi(\gamma)\ \hat{f}(\gamma)$$

then

$$|K_\phi(f)| \le ||\phi||_{\ell_{r'}(E)}\ \ ||\hat{f}|_E||_{\ell_r(E)}$$
$$\le C\ ||\phi||_{\ell_{r'}(E)}\ \ ||f||_{B_\alpha} \qquad \forall\ f \in B_\alpha$$

It follows that for some $g \in B_\alpha^{*}$,

$$K_\phi(f) = \int_G f(x)\ g(-x)\ dx \qquad \forall\ f \in B_\alpha$$

In particular, taking $f(x) = \gamma(x)$, $\gamma \in E$ we get $\hat{g}(\gamma) = \phi(\gamma)$ Hence $\phi \in (B_\alpha^{*}\hat{)}|_E$.

Conversely, suppose $\ell_{r'}(E) \subseteq (B_\alpha^{*})\hat{}$ Again, by the closed graph theorem, there exists a constant $C > 0$ such that

$$||g||_{B_\alpha^{*}} \le C\ ||\hat{g}||_{\ell_{r'}(E)} \qquad \forall\ \hat{g} \in \ell_{r'}(E)$$

Now let $f \in B_\alpha$ and put $\phi = \hat{f}|_E$ Define a linear functional on the class of functions on Γ with finite support by

$$K_\phi(\psi) = \sum_{\gamma\in E} \phi(\gamma)\ \psi(\gamma)$$

For each such ψ, there exists a trigonometric polynomial g with $\hat{g}(\gamma) = \psi(\gamma)$, so that

$$K_\phi(f) = \int_G f(x)\ g(-x)\ dx$$

and

$$\begin{aligned} |K_\phi(\psi)| &\le ||f||_{B_\alpha}\ ||g||_{B_\alpha^*} \\ &\le C||f||_{B_\alpha}\ ||\hat{g}||_{\ell_{r'}(E)} \\ &= C||f||_{B_\alpha}||\psi||_{\ell_{r'}(E)} \end{aligned}$$

Hence K_ϕ extends to a continuous linear functional on $\ell_{r'}(E)$ and so $\phi \in \ell_r(E)$.

The following lemma is essentially an interpolation result

<u>Lemma 2.5</u> Let E be a subset of Γ, $\beta > 0$ and $1 \le p,s < \infty$ Suppose there exists a constant $C > 0$ such that

$$||f||_{L^p} \le C\ p^\beta\ ||\hat{f}||_{\ell_s} \qquad \forall\ f \in \mathcal{T}_E$$

then if $0 < \alpha \le \beta$ and $q = p\beta/\alpha$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$||f||_{L^q} \le C_{\alpha,\beta}\ q^\alpha\ ||\hat{f}||_{\ell_r} \qquad \forall\ f \in \mathcal{T}_E$$

where $r = s'\beta/(s'\beta-\alpha)$

<u>Proof</u>: Consider the linear map U, defined on $\hat{\mathcal{T}}_E$ by

$$U\hat{f} = f \qquad \forall\ f \in \mathcal{T}_E$$

The hypothesis implies that U extends to a bounded linear map from

$\ell_s(E)$ to L^p_E with norm at most $C\ p^\beta$ Clearly U also extends to a bounded linear map from $\ell_1(E)$ to L^∞_E with norm at most 1 If $q=p\beta/\alpha$, let $\delta = 1-\alpha/\beta$ then $0 \le \delta < 1$ and $\frac{\delta}{1} + \frac{1-\delta}{s} = \frac{s'\beta-\alpha}{s'\beta} = \frac{1}{r}$

By Riesz-Thorin convexity theorem, U extends to a bounded linear map from $\ell_r(E)$ to L^q_E with norm at most

$$(C\ p^\beta)^{\alpha/\beta} = C^{\alpha/\beta}(\frac{\alpha}{\beta})^\alpha\ q^\alpha = C_{\alpha,\beta}\ q^\alpha$$

This completes the proof of the lemma

<u>Theorem 2.6</u>: Let G be a compact abelian group and Γ its dual group. Let $E \subseteq \Gamma$ and $0 < \alpha \le \beta$ Then E is a $\Lambda_{\alpha.\beta}$ set if and only if there exists a constant C depending only on α and β and not on q such that

$$||f||_{L^q} \le C\ q^\alpha\ ||\hat{f}||_{\ell_{r'}}, \qquad \forall\ f \in \mathcal{T}_E\ ,\ \forall q \ge 2\beta/\alpha \qquad (2.7)$$

where $r = 2\beta/\alpha$ (then $r' = 2\beta/(2\beta-\alpha)$).

<u>Proof</u> : <u>Sufficiency</u> · From Lemma 2 4, E is a $\Lambda_{\alpha,\beta}$ set if and only if $\ell_{r'}(E) \subseteq (B^*_\alpha)\hat{}\,|_E$

Now suppose $E \subseteq \Gamma$ and (2.7) holds Let $\phi \in \ell_{r'}(E)$. Since $2 \le r < \infty$, we have $1 < r' \le 2$ and $\ell_{r'}(E) \subseteq \ell_2(E)$ Hence there exists a $f \in L^2(G)$ such that $\hat{f}(\gamma) = \phi(\gamma)$ if $\gamma \in E$ and $\hat{f}(\gamma) = 0$ if $\gamma \notin E$. We claim that $f \in B^*_\alpha$ Let $\lambda > 0$ and consider

$$\int_G \exp(\ \lambda^{1/\alpha}\ |f(x)|^{1/\alpha}\ )dx = \sum_{k=0}^{\infty} \frac{\lambda^{k/\alpha}}{k!} \quad ||f||^{k/\alpha}_{L^{k/\alpha}}$$

$$= (\ \sum_{k \le 2\beta} + \sum_{k > 2\beta}\ )\ \frac{\lambda^{k/\alpha}}{k!} \quad ||f||^{k/\alpha}_{L^{k/\alpha}}$$

Using (2.7) for $q = k/\alpha$ in the second summation, we get

$$\sum_{k>2\beta} \frac{\lambda^{k/\alpha}}{k!} \; ||f||^{k/\alpha}_{L^{k/\alpha}} \le \sum_{k>2\beta} \frac{\lambda^{k/\alpha}}{k!} \; [C\;(k/\alpha)^{\alpha}\; ||\hat{f}||_{\ell_{r'}}]^{k/\alpha}$$

$$= \sum_{k>2\beta} C^{k/\alpha} \; \frac{\lambda^{k/\alpha}}{\alpha^{k}} \; \frac{k^{k}}{k!} \; ||\hat{f}||^{k/\alpha}_{\ell_{r'}}$$

$$\le \sum_{k>2\beta} [\; C^{1/\alpha} \; \frac{\lambda^{1/\alpha}}{\alpha} \; e \; ||\hat{f}||^{1/\alpha}_{\ell_{r'}} \;]^{k}$$

which is finite for a suitable choice of λ i.e. λ such that the expression in the square bracket is < 1.

<u>Necessity</u>: First we note that (2.7) is equivalent to saying that there exists a constant C_1 depending only on α and β such that

$$||f||_{L^{k/\alpha}} \le C_1 \; (k/\alpha)^{\alpha} \; ||\hat{f}||_{\ell_{r'}} \quad \forall\, f \in \mathcal{T}_E \;,\; k \in \mathbb{N} \text{ and } k \ge 2\beta \quad (2.8)$$

Clearly (2.7) implies (2.8). On the other hand if (2.8) holds and $q \ge 2\beta/\alpha$. let m be the unique integer such that $\frac{m-1}{\alpha} < q \le \frac{m}{\alpha}$. Then

$$||f||_{L^q} \le ||f||_{L^{m/\alpha}} \le C_1 (m/\alpha)^{\alpha} \; ||\hat{f}||_{\ell_{r'}}$$

$$\le C_1 \; (q + 1/\alpha)^{\alpha} \; ||\hat{f}||_{\ell_{r'}}$$

$$\le C_1 \; (1 + 1/2\beta)^{\alpha} \; q^{\alpha} \; ||\hat{f}||_{\ell_{r'}}$$

$$= C \; q^{\alpha} \; ||\hat{f}||_{\ell_{r'}}$$

Now suppose $E \subseteq \Gamma$ is a $\Lambda_{\alpha,\beta}$ set and (2.7), or equivalently (2.8) does not hold. Then for each $n \in \mathbb{N}$, there exists $f_n \in \mathcal{T}_E$ and an integer $k_n \ge 2\beta$ such that if $q_n = k_n/\alpha$, we have

$$||f_n||_{L^{q_n}} > n \; (q_n)^{\alpha} \; ||\hat{f}_n||_{\ell_{r'}}$$

Let
$$g_n = \frac{f_n}{||\hat{f}_n||_{\ell_{r'}}} \text{. Then } ||g_n||_{L^{q_n}} \ge C_{\alpha} \; n \; k_n^{\alpha}$$

We now estimate the norm $||g_n||_{B_\alpha^*}$. From the definition of the B_α^* norm, there exists $\lambda_n > 0$ such that

$$||g_n||_{B_\alpha^*} + C_n > \frac{1}{\lambda_n}[1 + \int_G \exp(\lambda_n^{1/\alpha} |g_n(x)|^{1/\alpha} dx]$$

where $0 < C_n \le e^{2\alpha}$

If some subsequence of $\{\lambda_n\}$ tends to zero, we get a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $||g_{n_k}||_{B_\alpha^*} \longrightarrow \infty$

If not, then there exists a $\delta > 0$ such that $\lambda_n \ge \delta \ \forall\, n$ Then

$$||g_n||_{B_\alpha^*} + C_n > \frac{1}{\lambda_n} \int_G \exp(\lambda_n^{1/\alpha} |g_n(x)|^{1/\alpha}) \, dx$$

$$\ge \frac{1}{\lambda_n} \frac{\lambda_n^{q_n}}{k_n!} ||g_n||_{L^{q_n}}^{q_n}$$

$$\ge \frac{\lambda_n^{(q_n)-1}}{k_n!} (C_\alpha \, n \, k_n^\alpha)^{q_n}$$

$$\ge \frac{1}{\delta} (\delta \, C_\alpha \, n)^{q_n} \frac{(k_n)^{k_n}}{k_n!} \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

Hence $\ell_{r'}(E) \nsubseteq (B_\alpha^*)\hat{}|_E$, so that by Lemma 2 4, E cannot be a $\Lambda_{\alpha,\beta}$ set. This completes the proof of the theorem

As a consequence of Theorem 2.6 we get the following characterization of Sidon subsets of Γ

Corollary 2.9 Let E be a subset of Γ Then

$$E \text{ is a Sidon set if and only if } \hat{B}_{1/2}|_E \subseteq \ell_2(E)$$

Proof: Let E be a Sidon subset of Γ, then as is well known ([21],p 59), we have

$$\|f\|_{L^p} \le C\, p^{1/2}\, \|f\|_{L^2} \qquad \forall\ f \in \mathcal{T}_E \text{ and } \quad 2 < p < \infty \qquad (*)$$

and by Theorem 2 6, this is equivalent to $\hat{B}_{1/2}|_E \subseteq \ell_2(E)$ For the converse, Pisier [23] has shown that, every $E \subseteq \Gamma$ satisfying (*) is a Sidon set

As another corollary, we get the following result·

Corollary 2.10 $\Lambda_{\alpha_1\ \beta} \subseteq \Lambda_{\alpha_2\ \beta}$, $0 < \alpha_2 < \alpha_1 \le \beta$

Proof: Let $E \in \Lambda_{\alpha,\beta}$ Then by Theorem 2 6 we get $(s_1 = \frac{2\beta}{2\beta-\alpha_1})$

$$\|f\|_{L^p} \le C\, p^{\alpha_1}\, \|\hat{f}\|_{\ell_{s_1}} \qquad \forall\ f \in \mathcal{T}_E\ ,\ \forall\ p \ge 2\beta/\alpha_1$$

Now applying Lemma 2 5 with $s = s_1$, $\beta = \alpha_1$, and $\alpha = \alpha_2$ we get $(q = \frac{p\alpha_1}{\alpha_2}\ ,\ p \ge \frac{2\beta}{\alpha_1})$

$$\|f\|_{L^q} \le C\, q^{\alpha_2}\, \|\hat{f}\|_{\ell_{s_2}} \qquad \forall\ f \in \mathcal{T}_E$$

Hence for every $q \ge \frac{2\beta}{\alpha_2}$, we get

$$\|f\|_{L^q} \le C\, q^{\alpha_2}\, \|\hat{f}\|_{\ell_{s_2}} \qquad \forall\ f \in \mathcal{T}_E\ .$$

So that by Theorem 2 6, $E \in \Lambda_{\alpha_2,\beta}$

This completes the proof of the corollary

2.11 Examples of $\Lambda_{\alpha,\beta}$ sets :

(i) If $E \subseteq \Gamma$ is a Sidon set and $1 < p < \infty$, then

$$\begin{aligned}\|f\|_{L^p} &\le C\, p^{1/2}\, \|\hat{f}\|_{\ell_2} \qquad \forall\ f \in \mathcal{T}_E \\ &\le C\, p^{\beta}\, \|\hat{f}\|_{\ell_2} \qquad \text{if } \beta \ge 1/2\end{aligned}$$

Therefore by the Lemma 2 5 and the Theorem 2.6, E is a $\Lambda_{\alpha,\beta}$ set for all $\beta \ge 1/2$ and $0 < \alpha \le \beta$

(ii) Recall ([21],p 15) that a subset $E \subseteq \Gamma$ is called an asymmetric set if $1 \notin E$ and whenever $\gamma \in E$, $\gamma^2 \neq 1$, then $\gamma^{-1} \notin E$ E is called a dissociate set if $1 \notin E$ and for every finite subset $F \subseteq E$ and a mapping $m\ F \longrightarrow \{0, \pm 1, \pm 2\}$ such that $\prod_{\gamma \in F} \gamma^{m(\gamma)} = 1$, we have $\gamma^{m(\gamma)} = 1$, $\forall\ \gamma \in F$

If G is an infinite compact abelian group, then Γ always contains infinite dissociate sets ([21],p 21)

Now let E be an infinite dissociate set and $k \in \mathbb{N}$. Define

$E_k = \{\prod_{\gamma \in S} \gamma\ \ S$ is an asymmetric subset of $E \cup E^{-1}$ with $|S| = k\}$

Then ([21],p 65) there exists a constant $A_k > 0$ such that

$$||f||_{L^q} \le A_k\, q^{k/2} ||\hat{f}||_{\ell_2} \qquad \forall\ f \in \mathcal{T}_{E_k}, \quad 2 < q < \infty$$

Therefore, by Lemma 2 5 and Theorem 2 6, E_k is a $\Lambda_{\alpha,\beta}$ set for $\beta \ge k/2$ and $0 < \alpha \le \beta$

(iii)The examples of $\Lambda_{\alpha,\beta}$-sets given above require $\beta \ge 1/2$ We now show that for $\beta < 1/2$, $\Lambda_{\alpha,\beta}$ sets need not exist.

Let $G = \prod_A Z_p$, where A is an infinite index set and $\Gamma = \prod_A^* Z_p$ its dual group Suppose $E \subseteq \Gamma$ is an infinite subset and F a finite subset of E. The subgroup H of Γ generated by F has cardinality atmost $p^{|F|}$.Let $V = H^{\perp}$, then V is an open subgroup of G,so $m(V) > 0$

Put
$$h = \frac{\chi_V}{m(V)}$$

Then $\hat{h} = \chi_H$ and so $||\hat{h}|_E||_{\ell_r} \ge |F|^{1/r}$, $1 \le r < \infty$.

Next we estimate $||h||_{B_\alpha}$. By Plancherel Theorem

$$||h||^2_{L^2} = \frac{1}{m(V)} = ||\hat{h}||^2_{L^2} = |H|$$

Hence $m(V)\ |H| = 1$, and we have,

$$\begin{aligned}||h||_{B_\alpha} &\le 1 + e + \int_G |h(x)|\ (\log^+|h(x)|)^\alpha\, dx \\ &\le 4 + \frac{1}{m(V)} \int_V (\log^+|H|)^\alpha\, dx = 4 + (\log^+|H|)^\alpha \\ &\le C\ (\log^+|H|)^\alpha \\ &= C\ |F|^\alpha (\log p)^\alpha\end{aligned}$$

Taking $\beta < 1/2$ and $r = 2\beta/\alpha < 1/\alpha$, we have

$$\frac{||\hat{h}|_E||_{\ell_r}}{||h||_{B_\alpha}} \longrightarrow \infty \qquad \text{as } |F| \longrightarrow \infty$$

Hence $\hat{B}_\alpha|_E \notin \ell_r(E)$, which proves that E is not a $\Lambda_{\alpha\,\beta}$ set

2.12 Now we include an application of the preceding results to some multiplier problems We observe that the following table is true

$$\begin{array}{llll} \quad\mathrm{I} & \quad\mathrm{II} & \quad\mathrm{III} & \quad\mathrm{IV} \\ \ell_{\frac{rq}{r-q}} & \subseteq (A_r\ ,\ \ell_q) & \subseteq ((B_\alpha)_r\ ,\ \ell_q) & \subseteq (L^p_r\ ,\ \ell_q),\ 1\le q<r<\infty, r>2 \\ \hat{M}(G) + \ell_{\frac{2r}{r-2}} & \subseteq (A_r\ ,\ A_r) & \subseteq ((B_\alpha)_r\ ,\ A_r) & \subseteq (L^p_r\ ,\ A_r),\quad r > 2 \\ \hat{M}_{\frac{qr}{r-q}}(G) + \ell_{\frac{2r}{r-2}} & \subseteq (A_r\ ,\ A_q) & \subseteq ((B_\alpha)_r\ ,\ A_q) & \subseteq (L^p_r\ ,\ A_q).\quad 2<q<r<\infty \end{array}$$

In Chapter 3, we mentioned that it is not known if the inclusion of I in II is proper. In Chapter 4, we proved that the inclusion of I in IV is proper. Using $\Lambda_{\alpha,\beta}$ sets, we show that the inclusion of I in III is proper if $\alpha \ge 1/2$. Also we prove that if $\alpha < 1/2$, $1 \le q < r < \infty$ and $r > 1/\alpha$, then $\ell_{\frac{rq}{r-q}} \subsetneq ((B_\alpha)_r\ ,\ \ell_q)$.

Theorem 2.13. Let $\alpha \geq 1/2$, then

(a) $\ell_{\frac{rq}{r-q}} \subsetneq ((B_\alpha)_r , \ell_q)$, $r > 2$, $1 \leq q < r < \alpha$

(b) $\hat{M}(G) + \ell_{\frac{2r}{r-2}} \subsetneq ((B_\alpha)_r , A_r)$, $r > 2$

(c) $\hat{M}_{\frac{rq}{r-q}}(G) + \ell_{\frac{2r}{r-2}} \subsetneq ((B_\alpha)_r , A_q)$, $2 < q < r < \alpha$

Also if $\alpha < 1/2$, then

(d) $\ell_{\frac{rq}{r-q}} \subsetneq ((B_\alpha)_r , \ell_q)$, $r > 1/\alpha$, $1 \leq q < r < \alpha$

Proof: Let E be an infinite $\Lambda_{1/2,1/2}$ set Then

$$\hat{B}_{1/2}|_E \subseteq \ell_2(E)$$

Case $\alpha \geq 1/2$ Then

$$\ell_{\frac{2q}{2-q}}(E) \subseteq (B_{1/2} , \ell_q) \subseteq ((B_{1/2})_r , \ell_q) \subseteq ((B_\alpha)_r , \ell_q) \text{ for } 1 \leq q < 2$$

and

$$\ell_\alpha(E) \subseteq (B_{1/2} , \ell_q) \subseteq ((B_{1/2})_r , \ell_q) \subseteq ((B_\alpha)_r , \ell_q) \text{ for } q \geq 2.$$

Now the proof of (a), (b), (c) follows exactly in the same manner as that of Theorem 3 1 of Chapter IV

Case $\alpha < 1/2$. Let E be an infinite $\Lambda_{\alpha,1/2}$ set Then

$$\hat{B}_\alpha|_E \subseteq \ell_{1/\alpha}(E)$$

Therefore

$$\ell_{\frac{q}{1-\alpha q}}(E) \subseteq (B_\alpha , \ell_q) \subseteq ((B_\alpha)_r , \ell_q) \text{ for } q < 1/\alpha$$

and

$$\ell_\alpha(E) \subseteq (B_\alpha , \ell_q) \subseteq ((B_\alpha)_r , \ell_q) \text{ for } q \geq 1/\alpha$$

Since $r > 1/\alpha$, $\ell_{\frac{rq}{r-q}} \subsetneq \ell_{\frac{q}{1-\alpha q}}$. Hence (d) follows

§ 3

In section 2, we gave a characterization of $\Lambda_{\alpha,\beta}$ sets It is easy to see that if $\beta_1 < \beta_2$ then every $\Lambda_{\alpha.\beta_1}$ set is also a Λ_{α,β_2} set. It is natural to ask whether the class of Λ_{α,β_1} sets is a proper subclass of Λ_{α,β_2} sets In this section we show that for each positive integer k there exists a subset $E \subseteq \Gamma$ which is a $\Lambda_{\alpha,k/2}$ set for $0 < \alpha \leq k/2$ but not a $\Lambda_{\alpha,\beta}$ set for any $\beta < k/2$

The main result of this section is Theorem 3 1 from which the above result about $\Lambda_{\alpha,\beta}$ sets follows as an immediate consequence. We would like to mention that the problem of deciding whether the classes of $\Lambda_{\alpha\ \beta}$ sets are distinct for distinct indices α remains unresolved

<u>Theorem 3.1</u>: Let G be an infinite compact abelian group and $k \in \mathbb{N}$. Then there exists a subset $E_k \subseteq \Gamma$ and a constant C_k such that

(i) $||f||_{L^q} \leq C_k\, q^{k/2}\, ||\hat{f}||_{\ell_2} \quad \forall\ f \in \mathcal{T}_{E_k}$ and $2 < q < \infty$

(ii) $\hat{B}_\alpha|_{E_k} \notin \ell_r\,(E_k), \quad 0 < \alpha \leq k/2$ and $r < k/\alpha$

<u>Corollary 3.2</u>. Let $k \in \mathbb{N}$. Then there exists a subset $E \subseteq \Gamma$ which is a $\Lambda_{\alpha,k/2}$ set for $0 < \alpha \leq k/2$, but not a $\Lambda_{\alpha.\beta}$ set for $0<\alpha\leq\beta<k/2$.

<u>Proof</u> : This is an immediate consequence of Theorems 2.6 and 3.1.

In addition, we now get the result mentioned in the introduction.

Corollary 3.3· Let G be an infinite compact abelian group, E a Sidon subset of Γ, and $0 < \alpha \le 1/2$ Then $\hat{B}_\alpha|_E \subseteq \ell_{1/\alpha}(E)$ Further, there exists a Sidon subset $E \subseteq \Gamma$ for which $\hat{B}_\alpha|_E \nsubseteq \ell_r(E)$ for every $r < 1/\alpha$

Proof: If E is a Sidon subset, then it is a $\Lambda_{\alpha,1/2}$ subset (example 2 11(i)) The set E constructed in Theorem 3 1 for $k = 1$ is a Sidon subset (cf cor 2 9) and $\hat{B}_\alpha|_E \nsubseteq \ell_r(E)$ for $r < 1/\alpha$ by (ii)

Proof of Theorem 3 1 :There are several steps in the proof of the above theorem We first prove the theorem in three special cases, namely, when $\Gamma = Z$, $Z(p^\alpha)$ and $\prod^*_{\alpha\in A} Z_{q_\alpha}$. where A is an infinite set Then we prove three lemmas and finally using these and the structure theorem for compact abelian groups, we reduce the general case to the above three special cases

In the proof for each of the three groups mentioned above, we start with a dissociate set E, so that (cf example (ii) §2) the set $E_k = E + E + \ldots + E$ (k- times) satisfies inequality (i) of the theorem. Next we construct a sequence $\{h_n\}$ of trigonometric polynomials such that

$$\frac{||h_n||_{B_\alpha}}{||\hat{h}_n|_{E_k}||_{\ell_r}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \alpha$$

where $1 \le r < k/\alpha$ and $0 < \alpha \le k/2$ Conclusion (ii) then follows from the closed graph theorem

3.4 The case $\Gamma = \mathbb{Z}$.

Let $E = \{3^{\ell}\}_{\ell=1}^{\infty}$. Then E is a Hadamard set with Hadamard constant 3, hence E is a dissociate set ([21], p 23). As remarked above, the set

$E_k = \{3^{\ell_1} + 3^{\ell_2} + \cdots + 3^{\ell_k} : \ell_j \in \mathbb{N},\ j = 1,2,\ldots,k\}$

satisfies inequality (i) with a constant depending only on k.

Let K_n denote the n-th Fejer kernel. For n large enough (so that $K_n(t) \le 1/2$ if $|t| \ge \pi/2$) we have

$$\|K_n\|_{B_\alpha} \le 1 + \pi e + 2\int_0^{\pi/2} K_n(x)\,(\log^+ K_n(x))^\alpha\,dx$$

Since $K_n(x) \le \min\left(n+1, \dfrac{\pi^2}{(n+1)x^2}\right)$, we get

$$\|K_n\|_{B_\alpha} \le 1 + e\pi + 2\int_0^{1/n+1} (n+1)\,(\log(n+1))^\alpha\,dx$$
$$+ 2\int_{1/n+1}^{\pi/2} \frac{\pi^2}{(n+1)x^2}\left(\log\frac{\pi^2}{(n+1)x^2}\right)^\alpha dx$$

$$\le 1 + e\pi + 2(\log(n+1))^\alpha + 2\frac{\pi^2}{(n+1)}(\log\pi^2(n+1))^\alpha (n+1-2/\pi)$$
$$\le C_\alpha (\log(n+1))^\alpha$$

Now let $h_n = K_{3^n k}$. Then for n large

$$\|h_n\|_{B_\alpha} \le C(\log 3^n k)^\alpha = C_k\, n^\alpha$$

To estimate $\|\hat{h}_n|_{E_k}\|_{\ell^r}$ we need an upper bound on $|[1, 3^n k] \cap E_k|$.

For this, we observe that if $m = 3^{\ell_1} + 3^{\ell_2} + \cdots + 3^{\ell_k}$ with $\ell_1 < \ell_2 < \cdots < \ell_k$, then the k-tuple $(\ell_1, \ell_2, \ldots, \ell_k)$ is uniquely determined by m. Hence

$$|[1, 3^n k] \cap E_k| \ge {}^nC_k \ge C_k\, n^k \quad \text{for n large enough.}$$

Therefore

$$||\hat{h}_n|_{E_k}||_{\ell_r} \geq C_k\ n^{k/r}$$

and if $1 \leq r < k/\alpha$,

$$\frac{||h_n||_{B_\alpha}}{||\hat{h}_n|_{E_k}||_{\ell_r}} \leq C_k\ n^{\alpha-k/r} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

<u>3 5 The case $\Gamma = Z(p^\infty)$</u> .

Recall that $Z(p^\infty) = \{\frac{j}{p^n} \cdot j \in Z,\ n \in Z\}/Z$, where p is a prime We write $[j/p^n]$ for the elements (equivalence classes) of $Z(p^\infty)$

In the following we assume $p \neq 2$ For $p = 2$ some modification is needed which we indicate at the end

Let $E = \{[1/p^n] .\ n \in Z\}$. Then E is a dissociate set, for if $\sum_{j=1}^{m} k_j [\frac{1}{p^{n_j}}] = 0$, where $n_1 < n_2 < \ldots < n_m$ and $k_j \in \{\pm 1, \pm 2\}$, then

$$\frac{k_m}{p^{n_m}} + \sum_{j=1}^{m-1} \frac{k_j}{p^{n_j}} \in Z$$

or

$$\frac{k_m\ p^{n_{m-1}}}{p^{n_m}} + p^{n_{m-1}} \sum_{j=1}^{m-1} \frac{k_j}{p^{n_j}} \in Z$$

But then, $\frac{k_m\ p^{n_{m-1}}}{p^{n_m}} \in Z$ which is impossible since $p \neq 2$.

Now fix $k > 0$ and let

$$E_k = E + E + \ldots + E \text{ (k-times)}$$

Then inequality (i) of the theorem holds for E_k (see example (ii) § 2).

Next let H_n be the subgroup generated by $[1/p^n]$ i e.

$$H_n = \{[j/p^n] :\ j = 1, 2, \ldots, p^n\}, \qquad |H_n| = p^n$$

Let $V_n = H_n^{\perp}$ and put $h_n = \dfrac{\chi_{V_n}}{m(V_n)}$. Then $\hat{h}_n = \chi_{H_n}$ We have seen in example (iii) § 2 that

$$m(V_n)\ |H_n| = 1 \text{ and } \qquad ||h_n||_{B_\alpha} \le C\ (\log p)^\alpha\ n^\alpha$$

We now estimate $||\hat{h}_n|_{E_k}||_{\ell_r} = |H_n \cap E_k|^{1/r}$

Clearly $|H_n \cap E_k| \le n^k$ If $m \in E_k$ has the representation

$$m = \frac{1}{p^{\ell_1}} + \frac{1}{p^{\ell_2}} + \qquad + \frac{1}{p^{\ell_k}} \quad \text{mod } Z, \text{ with } \ell_1 < \ell_2 < \quad < \ell_k ,$$

then the k-tuple $(\ell_1 ,\ell_2, \qquad ,\ell_k)$ is uniquely determined by m For, if not suppose

$$\frac{1}{p^{\ell_1}} + \frac{1}{p^{\ell_2}} + \qquad . + \frac{1}{p^{\ell_k}} = \frac{1}{p^{j_1}} + \qquad \frac{1}{p^{j_2}} + \qquad + \frac{1}{p^{j_k}} \quad \text{mod } Z$$

with $\ell_1 < \ell_2 < \qquad < \ell_k$ and $j_1 < j_2 <. \qquad < j_k$

Since both terms are less than 1, equality holds without mod Z Then after cancellation we may assume $\ell_1 < j_1$ Multiplying by p^{ℓ_1} we get,

$$1 + p^{\ell_1-\ell_2} + . \quad . + p^{\ell_1-\ell_k} = p^{\ell_1-j_1} + p^{\ell_1-j_2} +. \ . \ . + p^{\ell_1-j_k}$$

which is a contradiction since the left side is greater than 1, while the right is less than 1.

We conclude from this that $|H_n \cap E_k| \ge {}^nC_k \ge C_k\ n^k$ for large n, so that

$$C_k\ n^{k/r} \le ||\hat{h}_n|_{E_k}||_{\ell_r} \le n^{k/r}$$

and the conclusion (ii) of the theorem follows if $1 \le r < k/\alpha$.

For the case p = 2, we start with the set $E=\{[1/2^{2n}]:n \in Z\}$ Then the above proof with p replaced by 2^2 works in exactly the same way

3.6 The case $\Gamma = \prod^*_{\alpha\in A} Z_{q_\alpha}$

Here A is an infinite set, each q_α is a prime and $Z_{q_\alpha} = \{1, \omega_\alpha, \omega_\alpha^2, \ldots, \omega_\alpha^{q_\alpha - 1}\}$, where ω_α is a primitive q_α-th root of unity Γ is the dual of the compact abelian group $G = \prod_{\alpha\in A} Z_{q_\alpha}$

We consider two cases .

Case 1 $\sup_{\alpha\in A} q_\alpha < \infty$ In this case there exists a countably infinite set $\{\alpha_n\} \subseteq A$ such that for some prime p $q_{\alpha_n} = p$ for all n Let γ_n be a character on G defined by

$$\gamma_n(\omega) = \omega_{\alpha_n} , \qquad \omega \in G$$

Then each γ_n has order p Let $E = \{\gamma_n\}_{n=1}^{\infty}$ E is an independent set, hence a dissociate set Therefore the set $E_k = E + E + \quad + E$ (k-times) satisfies inequality (i) of Theorem 3 1

Let H_n be the subgroup generated by $\{\gamma_j\}_{j=1}^n$ and let $V_n = H^\perp$. Put

$$h_n = \frac{\chi_{V_n}}{m(V_n)} , \text{ then } \hat{h}_n = \chi_{H_n} \quad \text{and } |H_n| = p^n$$

By the same argument as in the case $\Gamma = Z(p^\infty)$, we see that

$$||h_n||_{B_\alpha} \le C_\alpha n^\alpha$$

and
$$C_k n^{k/r} \le ||\hat{h}_n|_{E_k}||_{\ell_r} \le n^{k/r}$$

and conclusion (ii) follows.

Case 2· $\sup_{\alpha\in A} q_\alpha = \infty$

Choose $\{q_{\alpha_n}\}$ such that $9k^2 < q_{\alpha_1} < q_{\alpha_2} < \quad < q_{\alpha_n}$ ..and $\lim_{n\to\infty} q_{\alpha_n} = \infty$.

As in case 1, define $\gamma_n(\omega) = \omega_{\alpha_n}$, $\omega \in G$ Then γ_n is of order q_{α_n} .

We let H_n be the subgroup generated by $\{\gamma_j\}_{j=1}^{n}$ and let $V_n = H_n^{\perp}$ and put

$$h_n = \frac{\chi_{V_n}}{m(V_n)}$$

Then

$$\hat{h}_n = \chi_{H_n}, \quad |H_n| = \prod_{j=1}^{n} q_{\alpha_j} \quad \text{and}$$

$$\|h_n\|_{B_\alpha} \le C\,(\log|H_n|)^{\alpha}$$
$$\le C\,(\log\prod_{j=1}^{n} q_{\alpha_j})^{\alpha}$$

In this case constructing the set E_k from the set $E = \{\gamma_n\}_{n=1}^{\infty}$ as in the earlier cases does not give the required estimate for $\|\hat{h}_n|_{E_k}\|_{\ell_r}$ We overcome this difficulty as follows

For each n, choose m_n as the largest integer for which $(3k)^{m_n+1} \le q_{\alpha_n}$ (where we fix $k \in \mathbb{N}$ as in the statement of the Theorem)

Let $$E = \{\gamma_n^{(3k)^j} \cdot n \in \mathbb{N} \quad j = 0, 1, 2, \ldots, m_n - 1\}$$

To see that E is a dissociate set, it is enough to show that for each n, the subset $\{\gamma_n, \gamma_n^{3k}, \ldots, \gamma_n^{(3k)^{m_n-1}}\}$ is a dissociate set, since the set $\{\gamma_n\}$ is independent. For this suppose

$$(\gamma_n)^{p_1(3k)^{\ell_1}} (\gamma_n)^{p_2(3k)^{\ell_2}} \cdots (\gamma_n)^{p_k(3k)^{\ell_m}} = 1 \text{ with } p_j = \pm 1, \pm 2$$

and $0 \le \ell_1 < \ell_2 \ldots < \ell_k < m_n$ Then, since p_1 is not divisible by 3,

$$p_1(3k)^{\ell_1} + \ldots + p_m(3k)^{\ell_m} = (3k)^{\ell_1}(p_1 + p_2(3k)^{\ell_2-\ell_1} + \ldots + p_m(3k)^{\ell_m-\ell_1}) \ne 0$$

Also $$\left|\sum_{j=1}^{k} p_j(3k)^{\ell_j}\right| \le 2\sum_{j=0}^{m_n-1} (3k)^j < (3k)^{m_n} < q_{\alpha_n}$$

so that

$$(\gamma_n)^{\sum_1^k p_j (3k)^{\ell_j}} \ne 1, \quad \text{a contradiction}$$

Now if we take $E_k = E + E + \dots + E$ (k-times), inequality (i) of the theorem holds It remains to estimate $||\hat{h}_n|_{E_k}||_{\ell_r} = |H_n \cap E_k|^{1/r}$

It is clear that $|H_n \cap E_k| \le {}^{\sum_1^n m_j}C_k$

We actually show that $|H_n \cap E_k| = {}^{\sum_1^n m_j}C_k$ For this, we show that each of the elements $(\gamma_{j_1})^{(3k)^{\ell_1}} (\gamma_{j_2})^{(3k)^{\ell_2}} \dots (\gamma_{j_k})^{(3k)^{\ell_m}}$ (where $1 \le j_1, \dots j_k \le n$ and $0 \le \ell_i < m_{j_i}$) are distinct

This will follow, if we show that $(\gamma_j)^{\sum_1^m s_i (3k)^{\ell_i}}$ never equals 1, where s_i and m are integers such that $0 < |s_i| \le 2k$ and $1 \le m \le 2k$ and $0 \le \ell_1 < \ell_2 < \dots < \ell_m < (3k)^{m_j}$

It is enough to show that $0 < |s_1(3k)^{\ell_1} + \dots + s_m(3k)^{\ell_m}| < q_{\alpha_j}$.

Now

$$|s_1(3k)^{\ell_1} + \dots + s_m(3k)^{\ell_m}| < (2k)\,(2k)\,(3k)^{m_j - 1} < (3k)^{m_j+1} \le q_{\alpha_j}.$$

Next, suppose

$$s_1(3k)^{\ell_1} + \dots + s_m(3k)^{\ell_m} = 0,$$

Then $$s_1 = -(3k)^{\ell_2 - \ell_1}\{s_2 + s_3(3k)^{\ell_3 - \ell_2} + \dots + s_m(3k)^{\ell_m - \ell_2}\},$$

which is not possible since $0 < |s_1| \le 2k$.

Thus we have seen that $|H_n \cap E_k| = {}^{\sum_1^n m_j}C_k \ge C_k \left(\sum_1^n m_j\right)^k$ for large n

$$\ge C_k' \left(\log \left(\prod_{j=1}^n q_{\alpha_j}\right)\right)^k.$$

Therefore

$||\hat{h}_n|_{E_k}||_{\ell_r} \geq C\,(\log(\prod_{j=1}^{n} q_{\alpha_j})^{k/r}$, which is the required estimate

<u>3.7</u> Finally to complete the proof of the Theorem 3 1 for a compact abelian group We need three short lemmas We shall need the following definitions ([21], p 24)

<u>3.8 Definitions</u>

Let s be a nonnegative integer, E_0 a subset of Γ and $\psi \in \Gamma$ $R_s(E_0,\psi)$ denotes the number (possibly ∞) of asymmetric subsets S of $E = E_0 \cup E_0^{-1}$ satisfying $|S| = s$ and $\prod_{\gamma\in S} \gamma = \psi$

Note that $R_s(E_0,\psi) = R_s(E,\psi)$ for all s and ψ

A subset $E_0 \subseteq \Gamma$ is called a <u>Rider set</u> if there exists a constant $B > 0$ such that $R_s(E_0,1) \leq B^s$ for all s. Note that if E_0 is a dissociate set, then E_0 is a Rider set since $R_s(E_0,1) = 0$ for all $s \geq 1$

If E is a Rider set, then inequality (i) of the Theorem 3 1 holds for E_k ([21], p 65)

For convenience, we will say that the group G has <u>property</u> P_k if there exists a Rider set $E \subseteq \Gamma$ such that $\hat{B}_\alpha|_{E_k} \notin \ell_r(E_k)$ for $0 < \alpha \leq k/2$ and $r < k/\alpha$.

Note that if a group G has property P_k, then the Theorem 3.1 holds for G

Lemma 3.9: Let H be a closed subgroup of G Suppose G/H has property P_k , then G has property P_k

Proof: By the hypothesis there exists a Rider subset $E \subseteq (G/H)^{\wedge} = H^{\perp}$ and a function $g \in B_{\alpha}(G/H)$ such that $\hat{g}|_{E_k} \notin \ell_r(E_k)$ We will show that the same set works for G also

Since $g \in B_{\alpha}(G/H)$, $go\pi \in B_{\alpha}(G)$, where $\pi.\ G \longrightarrow G/H$ is the quotient map. Also $(go\pi)^{\wedge}(\gamma) = \hat{g}(\gamma)$ for $\gamma \in H^{\perp}$

Therefore
$$(go\pi)^{\wedge}|_{E_k} \notin \ell_r(E_k)$$
and the lemma is proved

Lemma 3 10· Let $\{G_t\}_{t\in A}$ be a family of infinite compact abelian groups and $G = \prod_{t\in A} G_t$ If for some $t_o \in A$, G_{t_o} has property P_k , then G has property P_k.

Proof: We write $x = (x_t)_{t\in A}$ for elements of G with $e = (e_t)_{t\in A}$ as the identity element, and $\gamma = (\gamma_t)_{t\in A}$ for elements of the dual group Γ By the hypothesis, there exists a Rider set $E \subseteq \Gamma_{t_o}$ such that
$$B_{\alpha}(G_{t_o})^{\wedge}|_{E_k} \nsubseteq \ell_r\,(E_k)$$
Let $F = \prod_{t\neq t_o} e_{t_o} \times E$. Then it is easy to see that F is a Rider subset of Γ

If $g \in B_{\alpha}(G_{t_o})$ is such that $\hat{g}|_{E_k} \notin \ell_r(E_k)$, define $g_1(x) = g(x_{t_o})$, $x \in G$. Then $g_1 \in B_{\alpha}(G)$ and $\hat{g}_1(\gamma) = \hat{g}(\gamma_{t_o})$. Hence $\hat{g}_1|_{F_k} \notin \ell_r(F_k)$.

Lemma 3.11: Let H be an open subgroup of G. If there exists a dissociate subset $\tilde{E} \subseteq \Gamma/H^{\perp}$ such that $B_{\alpha}(H)^{\wedge}|_{\tilde{E}_k} \nsubseteq \ell_r(\tilde{E}_k)$, then G has property P_k .

Proof: Suppose $\tilde{E} = \{\tilde{\gamma}_\alpha \ \alpha \in A\} \subseteq \Gamma/H^\perp$ satisfies the hypothesis of the lemma Let γ_α be any representative of the coset $\tilde{\gamma}_\alpha$ for each $\alpha \in A$, and $E = \{\gamma_\alpha . \alpha \in A\}$ We claim that E is the required Rider set

If $\{\gamma_{\alpha_1}, \gamma_{\alpha_2}, \ \ , \gamma_{\alpha_n}\}$ is an asymmetric subset of $E \cup E^{-1}$ such that $1 = \gamma_{\alpha_1} \gamma_{\alpha_2} \ . \ \gamma_{\alpha_n}$. then $\tilde{1} = \tilde{\gamma}_{\alpha_1} \tilde{\gamma}_{\alpha_2} \ . \ \tilde{\gamma}_{\alpha_n}$ Since $\tilde{E}$ is a dissociate set $\tilde{\gamma}_{\alpha_j} = \tilde{1}$ for each $j = 1, 2, \ , n$. but this is not possible since $\tilde{1} \notin \tilde{E}$ Therefore $R_n(E, 1) = 0$ and so E is a Rider set

Now let $f \in B_\alpha(H)$ be such that $\hat{f}|_{\tilde{E}_k} \notin \ell_r(\tilde{E}_k)$ Define

$$g(x) = \begin{cases} f(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

Then $g \in B_\alpha(G)$ and

$$\begin{aligned} \hat{g}(\gamma_\alpha) &= \int_H (-x, \gamma_\alpha)\, g(x)\, dm_G(x) \\ &= C \int_H (-x, \gamma_\alpha)\, f(x)\, dm_H(x) \\ &= C\, \hat{f}(\gamma_\alpha) \end{aligned}$$

where $m_G|_H = C\, m_H$

Therefore $\hat{g}|_{E_k} \notin \ell_r(E_k)$. This completes the proof of the lemma

<u>3 12</u> <u>End of Proof of Theorem 3.1</u> : We consider two cases. First suppose Γ is not a torsion group, then G contains a closed subgroup H such that G/H is isomorphic to the circle group $\mathbb{T}$. By the Lemma 3 9 and the case $\Gamma = Z$, the theorem is true in this case

Now if Γ is a torsion group, then Γ is a weak direct product of p-primary groups ([10],A.3) By the Lemma 3 10 we may assume that Γ is a p-primary group Now there are two cases

Case 1. Γ is a p-primary divisible group. then Γ is a weak direct product of groups of the form $Z(p_{\alpha}^{\alpha})$ ([10], A14) By the Lemma 3 10 and the proof in the case $Z(p^{\alpha})$, the theorem holds in this case

Case 2 Γ is a p-primary non-divisible group Then Γ contains a subgroup $B = \prod_{\alpha}^{*} Z_{q_{\alpha}}$ such that Γ/B is divisible ([10], A 24) Now $B = (G/B^{\perp})^{\wedge}$, so that if B is infinite the theorem holds for $G/B^{\perp}$ hence also for G, by the Lemma 3 8

Finally if B is finite, then $B^{\perp}$ is an open subgroup of G and $(B^{\perp})^{\wedge} = \Gamma/B$ which is a divisible p-primary group By case 1 above the theorem holds for $B^{\perp}$. Using the Lemma 3 9 and the proof for the case $Z(p^{\alpha})$, we have in fact a dissociate set $\tilde{E} \subseteq \Gamma/B$ such that $B_{\alpha}(B^{\perp})^{\wedge}|_{\tilde{E}_k} \nsubseteq \ell_r(\tilde{E}_k)$ By the Lemma 3 11, there exists a Rider set $E \subseteq \Gamma$ such that $B_{\alpha}(G)^{\wedge}|_{E_k} \nsubseteq \ell_r(E_k)$ and the theorem holds for G.

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